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ERGODIC THEOREMS FOR A SEQUENCE OF MAPPINGS IN BANACH SPACES

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Abstract. In this paper, ergodic convergence for a sequence of nonexpansive mappings in uniformly convex and uniformly smooth Banach spaces is studied. By introducing another definition of an almost orbit, an attempt is made to replace the uniform convergence condition with a weaker one. Our results provide a generalization of some classical ergodic convergence theorems for nonexpansive mappings.

1. Introduction

Nonlinear ergodic theory for nonexpansive mappings is one of the fundamental topics in nonlinear functional analysis, closely related to fixed point theory and iterative methods. Besides its theoretical significance, this field has an important role in studying fixed point approximation algorithms, optimization problems, and analysing the asymptotic behavior of evolution systems of monotone type. A mapping $T : C \to C$ is called nonexpansive if for all x and y in C, we have: $\|Tx - Ty\| \leq \|x - y\|$. An ergodic theorem for the mapping T is a theorem that, under suitable conditions, asserts the Cesàro

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mean convergence of the iterates of the mapping T on an arbitrary point $x \in C$, i.e.,

$$\frac{1}{n}\sum_{k=0}^{n-1} T^k x.$$

Von Neumann's Mean Ergodic Theorem[18] is the starting point for this line of research in Hilbert spaces for linear nonexpansive mappings.

Theorem 1.1 (Von Neumann Ergodic Theorem)**.** [18] *Assume that T is a linear contraction mapping on a real Hilbert space H, that is, for every $x \in H$ we have $\|Tx\| \leq \|x\|$. Then, for every $x \in H$, $\frac{1}{n}\sum_{k=0}^{n-1} T^k x$ converges strongly to Px, where the mapping $P : H \to \mathrm{Fix}(T)$ is an orthogonal projection of H, that is, for every x, $P^2x = PPx = Px$, of x onto the fixed point set of T, $(\mathrm{Fix}(T) = \{x \in H : Tx = x\}$, which is a closed linear subspace of H). That is, the Hilbert space H admits the orthogonal decomposition*

$$H = \mathrm{Fix}(T) \oplus \overline{\mathrm{Ran}(I-T)}$$

(the direct sum of the fixed point set of T and the closure of the range of $I - T$).

This theorem was generalized to uniformly convex Banach spaces by Birkhoff [7] and Riesz [17]. About 40 years later, Baillon[5] generalized this theorem to nonlinear mappings. He showed that for a nonexpansive mapping on a nonempty closed convex subset of a Hilbert space, the existence of a fixed point is equivalent to the weak ergodic convergence of the iterates to a fixed point.

Theorem 1.2 (Baillon Ergodic Theorem)**.** [5] *Assume that C is a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping. Then the following statements are equivalent:*

1. $\mathrm{Fix}(T) \neq \emptyset$.
2. *For every $x \in C$, the sequence $\{S_n\}$ defined by $S_n x := \frac{1}{n}\sum_{k=0}^{n-1} T^k x$, converges weakly.*

In this case, if $S_n x$ converges weakly to Px, then $P : C \to \mathrm{Fix}(T)$ is nonexpansive such that for every $n \in \mathbb{N}$, we have: $P^2 = P$ and $PT^n = T^n P = P$. Moreover, for every $x \in C$, we have: $Px \in \overline{co}(\{T^i x\})_{i \geq 0}$ (where $\overline{co}(\{T^i x\})$ denotes the closure of the set of all finite convex combinations of the elements $T^i x$.).

In [16], Reich generalized Baillon's result to uniformly convex Banach spaces with a Fréchet differentiable norm. A real Banach space X with norm $\|\cdot\|$ is called uniformly convex if for every $\epsilon \in (0, 2]$, there exists $\delta = \delta(\epsilon) > 0$ such that for any $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$, we have:

$$\left\|\frac{1}{2}(x+y)\right\| < 1 - \delta.$$

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Let X^* be the dual of X. We denote the duality pairing by $\langle \cdot, \cdot \rangle$ on $X \times X^*$ and the unit sphere of X by S_X. The norm of X is said to be Gâteaux (resp. Fréchet) differentiable at $x \in S_X$ if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each point $y \in S_X$ (resp. uniformly for $y \in S_X$). A Banach space is smooth (resp. uniformly smooth) if the norm is Gâteaux differentiable on S_X (resp. Fréchet differentiable for all $x \in S_X$). If X is smooth, then the duality mapping J of X into X^* defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

is single valued.

A simpler proof of the nonlinear ergodic theorem was presented by Bruck in [9]. He first showed that any nonexpansive mapping on a nonempty closed convex bounded subset of a uniformly convex Banach space satisfies a property he called of type (γ) (see the next section for its definition). He then proved the ergodic theorem in uniformly convex Banach spaces with a Fréchet differentiable norm, not only for the iterates of the mapping but for any sequence that is its almost orbit. A sequence $\{x_n\}$ is an almost orbit of the mapping T if $\lim_{n\to\infty} \sup_{m\geq 0} \|x_{n+m} - T^m x_n\| = 0$.

Another point in Bruck's proof was that he proved not only the mean convergence but also the almost convergence of the almost orbits of the iterates. We say a sequence $\{x_n\}$ is strongly (weakly) almost convergent if $\frac{1}{n}\sum_{i=0}^{n-1} x_{i+k}$ is uniformly strongly (weakly) convergent uniformly in k. This type of convergence was first introduced by Lorentz [14]. Baillon also showed that a necessary and sufficient condition for the weak convergence of the sequence itself is that $T^{n+1}x - T^n x \rightharpoonup 0$.

In [11], Hirano presented another proof of Baillon's theorem in the same uniformly convex Banach spaces with a Fréchet differentiable norm. Moreover, in [12], he proved the ergodic theorem in uniformly convex spaces satisfying the Opial condition. Results concerning strong mean convergence under more restrictive assumptions on other mappings have also been investigated by several authors; for example, see [6]. Ergodic theorems in Hilbert and Banach spaces have also been studied for nonlinear nonexpansive semigroups. The interested reader may consult [6, 12, 15].

A one-parameter semigroup $S = \{S(t) : \ t \geq 0\}$ of nonexpansive mappings on C is a family of nonexpansive mappings from C into itself whenever the following conditions hold:

1. For every $s, t \geq 0$ and $x \in C$, we have: $S(s+t)x = S(s)S(t)x$.
2. For every $x \in C$, we have: $S(0)x = x$.
3. For $x \in C$, we have: $\lim_{t\to 0} S(t)x = x$.
4. For every $t \geq 0$ and $x, y \in C$, we have: $\|S(t)x - S(t)y\| \leq \|x - y\|$.

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These semigroups are continuous dynamical systems that describe the behavior of solutions of evolution equations associated with maximal monotone operators. The interested reader may consult [2, 3, 4].

The discrete dynamics generated by a nonexpansive mapping or the continuous dynamics generated by a nonexpansive semigroup are autonomous dynamics, that is, the evolution law is independent of time. However, if the evolution of the system depends on time, then in the discrete case we deal with a sequence of contractive mappings, and in the continuous case with a two-parameter evolution system $U = \{U(t, s) : t \geq s \geq 0\}$ of nonexpansive mappings on C satisfying the following conditions:

1. For every $t \geq s \geq r \geq 0$ and $x \in C$, we have: $U(t, s)U(s, r)x = U(t, r)x$.
2. For every $t \geq s \geq 0$ and $x, y \in C$, we have: $\|U(t, s)x - U(t, s)y\| \leq \|x - y\|$.
3. For every $t \geq 0$ and $x \in C$, we have: $U(t, t)x = x$.

The study of evolution systems is beyond the scope of this paper. Let us return, however, to the nonexpansive and nonautonomous system, namely a sequence $\{T_n\}$ of nonexpansive mappings from C into itself. Let $\{x_n\}$ and $\{z_n\}$ be sequences defined as follows:

$$x_{n+1} = T_n x_n, \ x_1 = x \in C \tag{1.1}$$

and

$$z_n = \frac{1}{n} \sum_{k=1}^{n} x_k. \tag{1.2}$$

Akatsuka et al. in [1] showed that in a real Hilbert space, if the sequence $\{T_n\}$ converges pointwise to T and

$$\mathrm{Fix}(T) \subset \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset, \tag{1.3}$$

then the sequence $\{z_n\}$ defined by (1.2) converges weakly to an element of $\mathrm{Fix}(T)$. This result is a generalization of the nonlinear ergodic theorem of Baillon for a sequence of nonexpansive mappings. These studies can be applied to investigate iterative methods involving nonexpansive mappings, such as the Mann iteration and the proximal point algorithm. The authors of [19] proved some ergodic theorems for sequences given by (1.1) and (1.2) in real Hilbert spaces by replacing the assumption (1.3) with the following weaker condition:

$$\sum_{n=1}^{\infty} \|T_n q - q\| < \infty. \tag{1.4}$$

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Kobayasi et al. in [13] obtained a weak almost convergence result for a sequence $\{x_n\}$ that satisfies the condition:

$$\lim_{n\to\infty} \sup_{m\geq 1} \|x_{n+m+1} - T_{n+m} \cdots T_n x_n\| = 0. \tag{1.5}$$

The condition (1.5) is clearly more general than (1.1). Such a sequence is called an almost orbit of the sequence $\{T_n\}$. We recall that the concept of an almost-orbit for the iterates of a nonlinear mapping was first introduced by Baillon [9]. An almost-orbit of a mapping T is defined as follows:

$$\lim_{n\to\infty} \sup_{m\geq 1} \|x_{n+m} - T_n^m x_n\| = 0. \tag{1.6}$$

The authors in [19] proved some similar ergodic theorems with this definition of an almost orbit. In the next section, we will prove the weak almost convergence for the sequence given by (1.6), and further show that the uniform convergence of T_n to T can be replaced by more general conditions. Furthermore, we replace the condition (1.4) with one of the following conditions:

$$\lim_{n\to\infty} \sup_{m\geq 1} \|T_{n+m} \cdots T_n q - q\| = 0 \tag{1.7}$$

or

$$\lim_{n\to\infty} \sup_{m\geq 1} \|T_n^m q - q\| = 0, \ \forall q \in \mathrm{Fix}(T). \tag{1.8}$$

Both conditions are weaker than (1.3), and the condition (1.7) is also weaker than (1.4).

2. Main Results

In [13], the authors proved the almost orbit convergence of a sequence given by (1.5), where $\{T_n\}$ is a sequence of nonexpansive mappings that converges uniformly to a nonexpansive mapping T in uniformly convex Banach spaces with a Fréchet differentiable norm, provided that condition (1.4) holds. It is easily seen that this result also holds by replacing the condition (1.4) with the more general condition (1.7) (see [13, Proposition 2.2]).

Definition 2.1. [9] *We denote by Γ the set of strictly increasing, continuous convex functions $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$. A mapping $T : C \to X$ is said to be of type (γ) if $\gamma \in \Gamma$ and for all $x, y \in C$ and $0 \leq \lambda \leq 1$,*

$$\gamma(\|\lambda Tx + (1-\lambda)Ty - T(\lambda x + (1-\lambda)y)\|) \leq \|x - y\| - \|Tx - Ty\|.$$

Lemma 2.2. [9] *Let C is a nonempty closed convex bounded subset of a uniformly convex real Banach space X, then there exists $\gamma \in \Gamma$ such that all nonexpansive mapping $T : C \to X$ is of type (γ).*

Lemma 2.3. [9] *Let C be a nonempty closed convex bounded subset of a uniformly convex real Banach space X. Let $T : C \to C$ be a mapping of type (γ) and $\{x_n\}$ be a sequence in C such that $\lim_{n\to\infty} \|x_{n+1} - Tx_n\| = 0$. Then for every weak neighborhood W of $\mathrm{Fix}(T)$, there exists a number N such that*

$$\frac{1}{n}\sum_{i=0}^{n-1} x_{i+k} \in W, \quad \forall n \geq N, \quad \forall k \geq 0.$$

Theorem 2.4. *Let C be a nonempty closed convex subset of a real uniformly convex and uniformly smooth Banach space X. Let $\{T_n\}$ be a sequence of nonexpansive mappings on C that converges uniformly to a nonexpansive mapping T on every bounded subset of C. If $\mathrm{Fix}(T) \neq \emptyset$ and the condition*

$$\lim_{n\to\infty} \sup_{m\geq 1} \|T_n^m q - q\| = 0, \ \forall q \in \mathrm{Fix}(T)$$

holds, then any sequence given by (1.6) is almost convergent to a fixed point of T.

Proof. Although the proof is similar to [13, Theorem 3.2], we present it step by step for the sake of completeness.

Step 1. Let $\{x_n\}$ and $\{y_n\}$ be sequences given by (1.6). Then $\|x_n - y_n\|$ is convergent. Let $\epsilon_n = \sup_{m\geq 0} \|y_{n+m+1} - T_n^m y_{n+1}\|$ and $\delta_n = \sup_{m\geq 0} \|x_{n+m+1} - T_n^m x_{n+1}\|$. Since $\{T_n\}$ is nonexpansive, we have:

$$\begin{aligned} \|x_{n+m+1} - y_{n+m+1}\| &\leq \|x_{n+m+1} - T^m x_{n+1}\| + \|T^m x_{n+1} - T^m y_{n+1}\| \\ &+ \|y_{n+m+1} - T^m y_{n+1}\| \\ &\leq \delta_n + \|x_{n+1} - y_{n+1}\| + \epsilon_n. \end{aligned}$$

By taking $\limsup$ as $m \to \infty$ and then $\liminf$ as $n \to \infty$, we get the desired result.

Step 2. If $\{x_n\}$ and $\{y_n\}$ satisfy condition (1.6) and $\{y_n\}$ is bounded, then for each $\lambda \in (0, 1)$, the sequence $\{\lambda x_n + (1-\lambda)y_n\}$ also satisfies condition (1.6). Define: $z_n = \lambda x_n + (1-\lambda)y_n$. Since $\{y_n\}$ is bounded, by Step 1, we have that $\{x_n\}$ is also bounded. Choose $r > 0$ such that $\{x_n\}, \{y_n\} \subset B(0, r)$ and also put $C_r = C \cap B(0, r)$ and $S_{n,m} = T_n^m\big|_{C_r}$, i.e., $S_{n,m}$ is equal to T_n^m restricted to C_r. Then C_r is a bounded closed convex subset and $S_{n,m} : C_r \to X$ is nonexpansive. Therefore, $S_{n,m}$ is of type γ.

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We have:

$$\begin{aligned}
\|z_{n+m+1} - S_{n,m}z_{n+1}\| &= \|\lambda x_{n+m+1} + (1-\lambda)y_{n+m+1} - S_{n,m}z_{n+1}\| \\
&\leq \lambda\|x_{n+m+1} - S_{n,m}x_{n+1}\| + (1-\lambda)\|y_{n+m+1} - S_{n,m}y_{n+1}\| \\
&+ \|\lambda S_{n,m}x_{n+1} + (1-\lambda)S_{n,m}y_{n+1} - S_{n,m}z_{n+1}\| \\
&\leq \lambda\alpha_n + (1-\lambda)\beta_n \\
&+ \gamma^{-1}(\|x_{n+1} - y_{n+1}\| - \|S_{n,m}x_{n+1} - S_{n,m}y_{n+1}\|) \\
&\leq \lambda\alpha_n + (1-\lambda)\beta_n \\
&+ \gamma^{-1}(\|x_{n+1} - y_{n+1}\| - \|x_{n+m+1} - y_{n+m+1}\| + \alpha_n + \beta_n)
\end{aligned}$$

where $\alpha_n = \sup_{m\geq 0}\|x_{n+m+1} - S_{n,m}x_{n+1}\|$ and $\beta_n = \sup_{m\geq 0}\|y_{n+m+1} - S_{n,m}y_{n+1}\|$. by Step 1 and (1.6), we obtain:

$$\lim_{n\to\infty}\sup_{m\geq 0}\|z_{n+m+1} - T_n^m z_{n+1}\| = 0.$$

Step 3. The sequence $\{\langle x_n, J(p-q)\rangle\}$ is convergent for all $p, q \in \mathrm{Fix}(T)$. If $q \in \mathrm{Fix}(T)$, then by (1.8), the constant sequence $\{q\}$ satisfies condition (1.6). Assume $0 < \lambda \leq 1$ and $p, q \in \mathrm{Fix}(T)$. By Step1 and Step2, the sequence $\{\|\lambda x_n + (1-\lambda)p - q\|\}$ is convergent as $n \to \infty$. Since $\{x_n - q\}$ is bounded, the norm differentiability of X implies that

$$\lim_{\lambda\to 0}\frac{1}{2\lambda}(\|q - p + \lambda(x_n - q)\|^2 - \|q-p\|^2) = \langle x_n - q, J(q-p)\rangle$$

uniformly in n. Therefore,

$$\lim_{n\to\infty}\langle x_n - q, J(q-p)\rangle = \lim_{n\to\infty}\lim_{\lambda\to 0}\frac{1}{2\lambda}(\|\lambda x_n + (1-\lambda)q - p\|^2 - \|q-p\|^2)$$

exists.

Step 4. Let $s_n = \frac{1}{n}\sum_{k=1}^{n} x_{k+i(n)}$, where $\{i(n)\}$ is a sequence of positive integers. It is clear that every weak limit point of s_n belongs to $\overline{co}\ \omega_w(\{x_n\})$ (where $\omega_w(\{x_n\})$ denotes the set of all weak limit points of $\{x_n\}$). On the other hand, our assumptions imply that

$$\|x_{n+1} - Tx_n\| \leq \|x_{n+1} - T_n x_n\| + \|T_n x_n - Tx_n\| \to 0$$

as $n \to \infty$. Now, by Lemmas 2.2 and 2.3 in [9], every weak limit point of s_n belongs to $\mathrm{Fix}(T) \cap \overline{co}\ \omega_w(\{x_n\})$. Let p and q be two weak limit points of s_n. Then $p, q \in \mathrm{Fix}(T)$ and by Step 3 we have that $\lim_{n\to\infty}\langle s_n - p, J(p-q)\rangle$ exists. Therefore $p = q$. This completes the proof. □

We have replaced the condition of the uniform convergence of T_n to T with a much weaker condition:

(2.1) $$\lim_{n\to\infty}\|\bar{x}_n - T\bar{x}_n\| = 0, \quad \forall \bar{x}_n \in \mathrm{Fix}(T_n).$$

Theorem 2.5. *Let C be a nonempty closed convex subset of a real uniformly convex and uniformly smooth Banach space X. Let $\{T_n\}$ be a sequence of nonexpansive mappings on C such that $\mathrm{Fix}(T_n) \neq \emptyset$, and let T be a mapping on C that satisfies condition (2.1). If (1.6) or*

$$w - \lim_{m\to\infty} \frac{1}{m}\sum_{i=0}^{m-1}(x_{n+i+k} - T_n^{i+k}x_n) = 0, \ \forall n \in \mathbb{N}, \ \textit{uniformly in } k \in \mathbb{N} \tag{2.2}$$

holds, then $\mathrm{Fix}(T) \neq \emptyset$ and $\{x_n\}$ is weakly almost convergent to $\bar{x} \in \mathrm{Fix}(T)$, which is the weak limit of $\{\bar{x}_n\}$.

Proof. Since T_n is nonexpansive and $\mathrm{Fix}(T_n) \neq \emptyset$, the weak ergodic theorem for nonexpansive mappings [9, 11] implies that $\frac{1}{m}\sum_{i=n}^{m} T_n^{i-n}x_n$ as $m \to \infty$, converges to $\bar{x}_n$, that is a fixed point of T_n. We first show that $\bar{x}_n$ is bounded. We have:

$$\begin{aligned}\|\frac{1}{m}\sum_{i=0}^{m-1} T_n^i x_n - \bar{x}\| \le \frac{1}{m}\sum_{i=0}^{m-1}\|T_n^i x_n - \bar{x}\| &\le \frac{1}{m}\sum_{i=0}^{m-1}\{\|T_n^i x_n - T_n^i\bar{x}\| + \|T_n^i\bar{x} - \bar{x}\|\} \\ &\le \|x_n - \bar{x}\| + \sup_{i\ge 0}\|T_n^i\bar{x} - \bar{x}\|.\end{aligned}$$

By taking $\liminf$ as $m \to \infty$, we have $\|\bar{x}_n - \bar{x}\| \le \|x_n - \bar{x}\| + \sup_{i\ge 0}\|T_n^i\bar{x} - \bar{x}\| < +\infty$. By (1.8) and the boundedness of $\{x_n\}$ (since $\mathrm{Fix}(T_n) \neq \emptyset$, each of the conditions (1.6) and (2.2) guarantees the boundedness of $\{x_n\}$), we conclude that $\{\bar{x}_n\}$ is bounded.

Therefore, the sequence $\{\bar{x}_n\}$ has a subsequence $\{\bar{x}_{n_j}\}$ that converges weakly to $\bar{x}$. The demiclosedness of T implies that $\bar{x} \in \mathrm{Fix}(T)$. Let $y \in X^*$ be arbitrary. We have:

$$\begin{aligned}\|\langle\frac{1}{m}\sum_{i=1}^{m} x_{i+k} - \bar{x}, y\rangle\| &= \|\langle\frac{1}{m}\sum_{i=1}^{n-1} x_{i+k} + \frac{1}{m}\sum_{i=n}^{m} x_{i+k} - \bar{x}, y\rangle\| \\ &= \|\frac{1}{m}\langle\sum_{i=1}^{n-1} x_{i+k}, y\rangle + \frac{1}{m}\langle\sum_{i=n}^{m} x_{i+k} - \bar{x}, y\rangle\| \\ &\le \frac{1}{m}\sum_{i=1}^{n-1}\|\langle x_i, y\rangle\| + \frac{1}{m}\|\sum_{i=0}^{m-n}\langle x_{i+k+n} - T_n^{i+k}x_n, y\rangle\| \\ &+ \frac{1}{m}\|\sum_{i=0}^{m-n}\langle T_n^{i+k}x_n - \bar{x}_n, y\rangle\| + \|\langle\bar{x}_n - \bar{x}, y\rangle\|.\end{aligned} \tag{2.3}$$

Now, letting $m \to \infty$, and using (2.2), we obtain:

$$\limsup_{m\to\infty}\|\langle\frac{1}{m}\sum_{i=1}^{m} x_{i+k} - \bar{x}, y\rangle\| \le \|\langle\bar{x}_n - \bar{x}, y\rangle\|.$$

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By substituting n with n_j and letting $j \to \infty$, the desired result is obtained. Also, using (2.3), we have:

$$\begin{aligned}\|\langle\frac{1}{m}\sum_{i=1}^{m} x_{i+k} - \bar{x}, y\rangle\| \leq& \frac{1}{m}\sum_{i=1}^{n-1}\|\langle x_{i+k}, y\rangle\| \\ &+ \|y\| \sup_{i\geq 0}\|x_{i+n} - T_n^i x_n\| \\ &+ \frac{1}{m}\|\sum_{i=0}^{m-n}\langle T_n^{i+k}x_n - \bar{x}_n, y\rangle\| + \|\langle \bar{x}_n - \bar{x}, y\rangle\|.\end{aligned}$$

letting $m \to \infty$, we have:

$$\limsup_{m\to\infty}\|\langle\frac{1}{m}\sum_{i=1}^{m} x_{i+k} - \bar{x}, y\rangle\| \leq \|y\| \sup_{i\geq 0}\|x_{i+n} - T_n^i x_n\| + \|\langle \bar{x}_n - \bar{x}, y\rangle\|.$$

By substituting n with n_j and letting $j \to \infty$, using (1.6), the desired result is obtained. □

Remark. In Theorem 2.5, the mappings T_n may be asymptotically nonexpansive mappings or any mappings for which the mean ergodic theorem holds, since the proof is based solely on the mean ergodic theorem for a single mapping. The assumption $\mathrm{Fix}(T_n) \neq \emptyset$ can be removed if we assume that C is bounded.

3. Conclusions

Our results are a generalization of some classical theorems on ergodic convergence for nonexpansive mappings. Therefore, the definition we have provided for an almost orbit can be used to generalize ergodic results (as we did in the past for Hilbert space [19] and in this paper for Banach spaces).

References

[1] M. Akatsuka, K. Aoyama and W. Takahashi, Mean ergodic theorems for a sequence of nonexpansive mappings in Hilbert spaces, *Sci. Math. Jpn.*, **68** (2008), no. 2, 233–239.

[2] F. Alvarez and J. Peypouquet, Asymptotic almost-equivalence of abstract evolution systems, *preprint submitted to Elsevier*, https://arxiv.org/pdf/0904.2157.

[3] F. Alvarez and J. Peypouquet, Asymptotic almost-equivalence of Lipschitz evolution systems in Banach spaces, *Nonlinear Anal.*, **73** (2010), no. 9, 3018–3033.

[4] F. Alvarez and J. Peypouquet, A unified approach to the asymptotic almost-equivalence of evolution systems without Lipschitz conditions, *Nonlinear Anal.*, **74** (2011), no. 11, 3440–3444.

[5] J.-B. Baillon, Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert, *(French) C. R. Acad. Sci. Paris Sér. A-B*, **280** (1975), 1511–1514.

[6] J. B. Baillon, R. E. Bruck and S. Reich, On the asymptotic behavior of nonexpansive mappings and semigroups in Banach spaces, *Houston J. Math.*, **4** (1978), no. 1, 1–9.

[7] G. D. Birkhoff, The mean ergodic theorem, *Duke Math. J.*, **5** (1939), no. 1, 19–20.

[8] F. E. Browder and W. V. Petryshyn, The solution by iteration of nonlinear functional equations in Banach spaces, *Bull. Amer. Math. Soc.*, **72** (1966), 571–575.

[9] R. E. Bruck, A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces, *Israel J. Math.*, **32** (1979), no. 2-3, 107–116.

[10] R. E. Bruck and S. Reich, Accretive operators, Banach limits, and dual ergodic theorems, *Bull. Acad. Polon. Sci. Sér. Sci. Math.*, **29** (1981), no. 11-12, 585–589.

[11] N. Hirano, A proof of the mean ergodic theorem for nonexpansive mappings in Banach space, *Proc. Amer. Math. Soc.*, **78** (1980), no. 3, 361–365.

[12] N. Hirano, Nonlinear ergodic theorems and weak convergence theorems, *J. Math. Soc. Japan*, **34** (1982), no. 1, 35–46.

[13] K. Kobayasi and I. Miyadera, Some remarks on nonlinear ergodic theorems in Banach spaces, *Proc. Japan Acad. Ser. A Math. Sci.*, **56** (1980), no. 3, 88–92.

[14] G. G. Lorentz, A contribution to the theory of divergent sequences, *Acta Math.*, **80** (1948), 167–190.

[15] O. Nevalinna and S. Reich, Stronge convergence of contraction semigroups and of iterative methods for accretive operators in Banach spaces, *Israel J. Math.*, **32** (1979), 44–58.

[16] S. Reich, Nonlinear ergodic theory in Banach spaces, *Argone Nat. Lab. Springfield*, **79(69)** (1979), pp. 1–22.

[17] F. Reisz, Some mean ergodic theorems, *J. London Math. Soc.*, **13** (1938), no. 4, 274–278.

[18] J. V. Neumann, Proof of the quasi-ergodic hypothesis, *Proc. Acad. USA*, **18** (1932), no. 1, 70–82.

[19] A. Yousefi Pirbasti, H. Khatibzadeh and M. R. Piranfar, Some convergence theorems for a sequence of nonexpansive mappings in a Hilbert space, *Rend. Circ. Mat. Palermo (2)*, **74** (2025), no. 6, Paper No. 183, 15 pp.

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