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DYNAMICAL SYSTEMS WITH STABLE STATISTICAL BEHAVIOR

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Abstract. This paper introduces a new class of endomorphisms on the two-dimensional torus $\mathbb{T}^2$ that defy classical structural classifications. Specifically, we demonstrate the existence of an open set of maps that are neither uniformly expanding nor admit any dominated splitting, yet exhibit robust statistical stability. We prove that these maps stably admit a finite number of ergodic absolutely continuous invariant probability measures (ACIPs). The proof relies on the concept of "virtually expanding" maps, recently introduced by M. Tsujii, which utilizes the spectral properties of the transfer operator. We provide a constructive proof using surgical perturbations of linear endomorphisms to generate explicit examples satisfying these conditions.

1. Introduction

The problem of rigorously describing the long-term statistical behavior of deterministic dynamical systems has been a central theme in mathematics since the early twentieth century. Its origins lie in the foundational works of Birkhoff and von Neumann, whose ergodic theorems established a rigorous link between time averages along individual trajectories and space averages with respect to invariant probability measures. These results provided a mathematical justification for the use of statistical laws in deterministic systems, famously bridging the gap between thermodynamics and mechanics. However, they left open a fundamental question: among the typically large and often uncountable

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family of invariant measures admitted by a dynamical system, which ones are relevant for describing the behavior of typical initial conditions from a physical or observational point of view?

This question became particularly pressing with the emergence of chaos theory and the recognition that deterministic systems may exhibit sensitive dependence on initial conditions. In such systems, individual trajectories are inherently unpredictable, yet numerical experiments and physical observations consistently reveal stable statistical patterns. This apparent paradox suggested the existence of "distinguished" invariant measures governing the macroscopic behavior of the system. These measures, often called *physical measures* or *SRB (Sinai-Ruelle-Bowen) measures*, are characterized by the property that the time averages of observables for a set of initial conditions with positive Lebesgue measure converge to the space average defined by the measure.

The first major success in identifying such measures came with the study of *uniformly hyperbolic* systems (Anosov diffeomorphisms and Axiom A attractors) in the 1960s and 70s. In these systems, the tangent space splits continuously into uniformly expanding and contracting directions. This rigid geometric structure ensures both structural stability and the existence of a unique (or finitely many) physical measures that are stable under perturbation.

However, uniform hyperbolicity is a very strong condition that is often not satisfied in physical models. This led to the study of *non-uniformly hyperbolic* systems, where expansion and contraction are only asymptotic or intermittent. While significant progress was made—such as the existence of absolutely continuous invariant probability measures (ACIPs) for certain one-dimensional maps and the Hénon map, these systems are often statistically fragile. A small perturbation can destroy the chaotic attractor or the physical measure.

To address this, the concept of *dominated splitting* (or partial hyperbolicity) was introduced. A dynamical system admits a dominated splitting if the tangent bundle can be decomposed into two invariant sub-bundles such that the expansion in one strictly dominates the expansion in the other. This condition is robust (C^1 open) and serves as a generalized geometric skeleton, allowing for the extension of many results from uniform hyperbolicity to a broader class of systems. For decades, dominated splitting has been the standard assumption for proving the robust existence of physical measures in higher dimensions.

The Central Problem. The research presented in this paper addresses a fundamental open question regarding the necessity of these geometric structures. **Is dominated splitting a necessary condition for robust statistical stability?** Can we find open classes of dynamical systems that lack both uniform expansion and dominated splitting, yet still possess a stable, finite family of physical measures?

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The prevailing intuition in the field suggests that without the separation of scales provided by dominated splitting, the mixing of expanding and contracting directions would be too chaotic and "tangled" to support a stable statistical distribution. This paper challenges that intuition. We investigate a class of endomorphisms (non-invertible maps) on the two-dimensional torus $\mathbb{T}^2$ and prove the existence of an open set of maps that violate all classical hyperbolic splitting conditions but still maintain ideal statistical properties.

Approach: Virtual Expansion. To achieve this, we move beyond the classical geometric analysis of the tangent bundle and employ the concept of **virtually expanding** maps, a notion recently introduced by M. Tsujii. This approach shifts the focus to the *cotangent bundle* and the spectral properties of the *transfer operator* (Perron-Frobenius operator). We show that statistical stability is governed by the action of the dynamics on densities, which can remain well-behaved (contracting) even when the action on individual vectors (geometry) is degenerate.

By constructing explicit examples of such maps through surgical perturbation, we demonstrate that "virtual expansion" is a distinct and sufficient mechanism for statistical stability, independent of dominated splitting. This work creates a bridge between abstract spectral theory and concrete dynamical constructions, expanding the known universe of statistically stable systems.

2. Main Results

In this section, we detail the theoretical framework of virtually expanding endomorphisms, provide the geometric criteria for their identification, and state the main theorem regarding the existence of the exotic class of maps on the torus.

2.1. The Concept of Virtual Expansion.

Classically, the chaotic behavior of a map $f : M \to M$ is analyzed by examining the derivative Df acting on the tangent bundle TM. Expansion corresponds to the growth of tangent vectors. However, when a system lacks dominated splitting, the directions of expansion and contraction may rotate and mix in complex ways, preventing the definition of invariant expanding cones.

Virtual expansion offers a dual perspective. Instead of tracking vectors in TM, we track co-vectors (linear functionals) in the cotangent bundle T^*M. The dynamics induce a "pull-back" map $f^\dagger$ on the cotangent bundle. If the map f is expanding in a generalized sense, the pull-back map on co-vectors should be contracting.

The rigorous definition relies on the **Transfer Operator** $\mathcal{L}$, which describes the evolution of probability densities under the dynamics. For a map f, the operator is defined as:

$$\mathcal{L}\varphi(x) = \sum_{f(y)=x} \frac{\varphi(y)}{|Jf(y)|}$$

where $|Jf(y)|$ is the Jacobian determinant. The system is said to be **virtually expanding** if this operator acts as a contraction on appropriate functional spaces, specifically Sobolev spaces of high regularity.

M. Tsujii defined a functional $b^\mu((x,\xi);f)$ that measures the "inverse expansion" accumulated along the backward orbits of a co-vector (x,ξ). A map is virtually expanding if the asymptotic growth rate of this functional is strictly less than 1. The crucial consequence of this definition is spectral:

- **Quasi-compactness:** For a virtually expanding map, the transfer operator $\mathcal{L}$ is quasi-compact on the Sobolev space $H^\mu(M)$. This means the essential spectral radius is bounded by some $\rho < 1$.
- **Discrete Spectrum:** The portion of the spectrum outside the disk of radius ρ consists of a finite number of eigenvalues with finite multiplicity.

2.2. **A Geometric Sufficient Condition.** While the definition of virtual expansion is analytical, verifying it for specific examples requires a geometric criterion. A significant contribution of this work is the derivation of a sufficient condition that depends only on the local behavior of the derivative and the Jacobian.

Lemma (Sufficient Condition): Let $f : M \to M$ be a C^r endomorphism. If there exists $\mu > 0$ such that for every $x \in M$:

$$\sum_{f(y)=x} \frac{1}{|Jf(y)| \cdot m(D_y f)^\mu} < 1$$

then f is $(\mu/2)$-virtually expanding.

Here, the sum is taken over all pre-images y of x. The term $|Jf(y)|$ represents the volume expansion, and $m(D_y f) = \|(D_y f)^{-1}\|^{-1}$ is the minimum expansion (co-norm) of the derivative at y.

This inequality reveals the flexibility of endomorphisms. Unlike diffeomorphisms, where a single "bad" point in the trajectory can ruin hyperbolicity, endomorphisms allow for local failures of expansion. A point x may have one pre-image y_1 where the derivative is not expanding (small $m(D_{y_1} f)$), provided it has other pre-images $y_2, \ldots, y_d$ where the map is strongly expanding or the Jacobian is very large. As long as the sum over all pre-images remains less than 1, the system is globally virtually expanding. This averaging mechanism is the key to constructing our counter-examples.

2.3. **Main Theorem.** Using the geometric condition above, we prove the existence of the open class of maps that is the subject of this paper.

Theorem A: There exists a non-empty, C^1-open set $\mathcal{U}$ of local diffeomorphisms on the torus $\mathbb{T}^2$ such that for any $f \in \mathcal{U}$:

(1) **Non-Uniformity:** The map f is not uniformly expanding. There exist regions in $\mathbb{T}^2$ where the derivative is not an expansion.

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(2) **No Dominated Splitting:** The map f does not admit any dominated splitting of the tangent bundle $T\mathbb{T}^2$. Specifically, there is no invariant decomposition $TM = E \oplus F$ such that expansion in one strictly dominates the other.
(3) **Stable Statistics:** Despite the lack of the above structures, f admits a finite number of ergodic absolutely continuous invariant probability measures (ACIPs).
(4) **Regularity:** The densities of these invariant measures belong to the Sobolev space $H^r(\mathbb{T}^2)$ for some $r > 0$.

This theorem affirmatively answers the question posed in the introduction. It demonstrates that "virtual expansion" is a distinct mechanism for statistical stability that can operate in the absence of the geometric skeleton provided by dominated splitting.

3. Summary of Proofs/Conclusions

The proof of the Main Theorem is constructive. We employ a technique known as "surgical perturbation" to modify a standard linear map into one with the desired exotic properties. This section outlines the steps of the construction and summarizes the conclusions.

3.1. **Construction Strategy. Step 1: The Linear Baseline** We begin with a linear endomorphism of the torus, $f_0(x) = Ax \pmod 1$, where A is a 2×2 integer matrix with determinant $d > 1$. We assume the eigenvalues of A are strictly outside the unit circle. Such a map is uniformly expanding and possesses a dominated splitting (unless the eigenvalues have equal modulus, in which case the splitting is trivial but the map is still expanding).

Step 2: Local Perturbation We define a small neighborhood V around a fixed point p of the map. We construct a new map f that coincides with the linear map Ax everywhere outside V. Inside V, we deform the map using a smooth bump function. The derivative of the perturbed map, Df, interpolates between the original matrix A (at the boundary) and a target matrix B (at the center of V).

Step 3: Breaking Dominated Splitting The core of the proof lies in the choice of the matrix B. To ensure the resulting map has *no* dominated splitting, B must be chosen to destroy the invariance of the cone fields associated with A. Dominated splitting implies that vectors in a "stable" cone always map to the stable cone and expand less than vectors in the "unstable" cone. We choose B to be a rotation or a matrix that mixes these directions. For example, if B rotates vectors by 90 degrees, it maps the unstable direction of A into the stable direction of A. By carefully analyzing the cone fields, we show that this local rotation makes it impossible to define a continuous invariant splitting over the entire manifold. Thus, the perturbed map f has no dominated splitting.

Step 4: Verifying Virtual Expansion While the perturbation destroys the geometric splitting, we must ensure it does not destroy the statistical stability. We apply the sufficient condition derived in Lemma 4.1:

$$\sum_{f(y)=x} \frac{1}{|Jf(y)| \cdot m(D_y f)^{\mu}} < 1$$

For any point x in the torus, there are $d = |\det A|$ pre-images.

- **Unperturbed Pre-images:** Since the perturbation is local and the map is a covering map of degree d, most pre-images of x will lie outside the perturbed neighborhood V. At these points, the map acts as the linear matrix A, which is strongly expanding. The terms in the sum corresponding to these points are very small.
- **Perturbed Pre-image:** At most one pre-image can lie inside V. At this point, the derivative is close to B, which might be non-expanding or contracting ($m(B)$ could be small). Consequently, this single term in the sum might be large.

However, the "virtual" nature of the expansion allows us to compensate. As long as the degree d (the determinant of A) is sufficiently large, the smallness of the $d-1$ unperturbed terms dominates the single large perturbed term. The sum remains strictly less than 1. This proves that f is virtually expanding.

Step 5: Robustness The property of having "no dominated splitting" is robust because we have broken the condition transversally; we are in the interior of the complement of the set of maps with dominated splitting. The virtual expansion condition is defined by a strict inequality (< 1). Therefore, any map g that is sufficiently close to f in the C^1 topology will also satisfy both conditions. This confirms the existence of the open set $\mathcal{U}$.

3.2. **Spectral Consequences.** Having established that maps in $\mathcal{U}$ are virtually expanding, we invoke Tsujii's theorem regarding the transfer operator. Since $f \in \mathcal{U}$ implies f is virtually expanding, the transfer operator $\mathcal{L}$ acts as a quasi-compact operator on the Sobolev space $H^r(\mathbb{T}^2)$. This spectral gap implies: 1. The existence of a finite number of ACIPs. 2. The densities of these measures are regular (elements of H^r).

3.3. **Conclusions.** This work provides a significant contribution to the classification of chaotic dynamical systems. By identifying a class of maps that are "virtually expanding" but lack traditional hyperbolic structures like dominated splitting, we broaden the scope of systems known to exhibit physical measures.

The results highlight the fundamental difference between diffeomorphisms and endomorphisms. In invertible systems (diffeomorphisms), structural stability and statistical stability are often tightly linked to uniform hyperbolicity or dominated splitting. In non-invertible systems (endomorphisms),

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the multiplicity of pre-images provides a mechanism for resilience. The system can tolerate local regions of "bad" geometry (contraction or rotation) because the global statistics are determined by the sum over all pre-images. If the "good" pre-images outweigh the "bad" ones, statistical stability is preserved.

This suggests that the condition of virtual expansion may be the more fundamental criterion for the existence of ACIPs in non-uniformly hyperbolic settings. This opens new avenues for research, particularly in exploring whether these conditions can be extended to higher-dimensional manifolds beyond the torus or to systems with critical points. The geometric criteria provided herein offer a practical tool for testing other dynamical systems for statistical stability, bridging the gap between abstract ergodic theory and concrete physical modeling.

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