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# COMPARISON THEOREMS AND THEIR APPLICATIONS ON RIEMANNIAN MANIFOLDS WITH BOUNDED RICCI CURVATURE

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Abstract. In this paper, under a lower bound condition on the Ricci curvature involving vector fields and gradient vector fields on a Riemannian manifold $M^n$, we investigate several comparison theorems. We first establish Laplacian comparison and volume comparison theorems for Riemannian manifolds endowed with the modified Ricci curvature

$$\tilde{Ric} := Ric + \frac{1}{2}\mathcal{L}_V g \geq (n-1)k,$$

and then for manifolds with Bakry–Émery Ricci curvature

$$Ric + Hessh \geq (n-1)k,$$

for some smooth function $h$. We further show that, in particular cases, these theorems hold for shrinking, steady, and expanding gradient Ricci solitons under a volume non-collapsing condition, and even without this condition when the soliton function is bounded. Consequently, we obtain analogous results for almost Ricci solitons and almost gradient Ricci solitons. As an application of these comparison results, we establish a segment inequality for Riemannian manifolds $M^n$ with bounded Ricci curvature.

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## 1. Introduction

Let $\gamma : (0, \infty) \longrightarrow M$ be a normal geodesic with initial point $p$. It is known that if $t > 0$ is sufficiently small, then $d(p, \gamma(t)) = t$, meaning that $\gamma([0, t])$ is a minimizing geodesic. If $\gamma([0, t_1])$ is not minimizing, then it is also non-minimizing for all $t > t_1$. Continuing this process, the set of positive numbers $t$ for which $d(p, \gamma(t)) = t$ holds is either of the form $[0, t_0]$ or $[0, \infty)$. In the first case, $\gamma(t_0)$ is called the cut point of $p$ along $\gamma$, and in the second case no such cut point exists. The set of cut points of $p$ is denoted by $cut(p)$.

Now consider a Riemannian manifold $(M, g)$ endowed with the volume element

$$dVol = \sqrt{G \circ x^{-1}}\, dx^1 \cdots dx^n,$$

defined on a chart $(x, U)$ of $M$, where $G = \det(g_{ij})$ and $dx^1 \cdots dx^n$ denotes the Lebesgue measure on $\mathbb{R}^n$. Fix a point $p \in M$. Since the exponential map $\exp_p$ is a diffeomorphism on the complement of the cut locus of $p$, one can construct a chart

$$\{\exp_p^{-1},\, M \setminus cut(p),\, \Sigma(p)\}$$

on $M$, where

$$\Sigma(p) = \exp^{-1}(M \setminus cut(p)).$$

In these coordinates, the volume element of $M$ admits the following decomposition:

$$dVol = \lambda(r, \theta)\, dr \wedge d\theta, \tag{1.1}$$

where $d\theta$ is the standard volume element on the sphere $S^{n-1}$ and $\lambda(r, \theta)$ is a smooth function on $\Sigma(p)$.

Since the volume element of a Riemannian manifold admits a decomposition of the form (1.1), this provides a motivation to compare the volume of Riemannian manifolds with the corresponding volume on manifolds of constant curvature $K$, which are called model spaces. Among the earliest results establishing a relationship between volume on a Riemannian manifold and volume on the model space are the Bishop–Gromov volume comparison theorems [5]. In fact, they compared the volumes of geodesic balls of radius $r$ in a complete Riemannian manifold with those in the model space and proved that the function

$$f : r \longmapsto \frac{Vol(B_r(p))}{Vol(B_r^K)}$$

is non-increasing and satisfies $\lim_{r \to 0} f(r) = 1$. In particular, they showed that

$$Vol(B_r(p)) \geq Vol(B_r^K),$$

where $B_r(p)$ denotes the geodesic ball of radius $r$ centered at $p$ in the Riemannian manifold and $B_r^K$ is the geodesic ball of radius $r$ in the model space $M_K^n$. Consequently, Bishop and Gromov obtained

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an upper bound for the volume growth:

$$Vol(B_r(p)) \geq Vol(B_r^K) = w_n r^n,$$

where $w_n$ denotes the volume of the unit ball in $\mathbb{R}^n$, and equality holds if and only if $(M, g)$ is isometric to $(\mathbb{R}^n, g_0)$. This result provided strong motivation for mathematicians to address the following question: what is the largest exponent $\alpha$ such that

$$Vol(B_r(p)) \geq cr^{\alpha}?$$

In order to address this question, we first recall the following definitions.

Let $(M, g)$ be a Riemannian manifold. The Ricci curvature tensor is defined for any $X, Y \in T_pM$ by

$$Ric(X, Y) := g^{il} g(R(\partial_i, X)Y, \partial_l).$$

In local coordinates, the components of this tensor are usually denoted by $R_{jk}$ and are given by

$$R_{jk} = g^{il} R_{ijkl}.$$

A Riemannian manifold $(M, g)$ is called a Ricci soliton if and only if there exists a smooth vector field $V$ such that

$$Ric(g) = \lambda g - \frac{1}{2}\mathcal{L}_V g,$$

for some constant $\lambda \in \mathbb{R}$, where $Ric$ denotes the Ricci curvature tensor and $\mathcal{L}$ is the Lie derivative.

Calabi and Yau proved that for complete noncompact Riemannian manifolds with nonnegative Ricci curvature (Ric $\geq 0$), there exists a constant $c$ such that for all $r > 2$,

$$Vol(B_r(p)) \geq cr.$$

For further background on comparison theorems, see [6, 8, 14, 16].

(Almost Ricci Soliton) Let $V$ be a smooth vector field on a Riemannian manifold $(M, g)$. A Ricci soliton on this manifold is defined by

$$\text{Ric} + \frac{1}{2}\mathcal{L}_V g = \lambda g, \tag{1.2}$$

where $\mathcal{L}_V g$ denotes the Lie derivative of the metric, Ric is the Ricci curvature tensor of $M$, and $\lambda \in \mathbb{R}$ is a constant. In fact, Ricci solitons are self-similar solutions to the Ricci flow equation. A Ricci soliton is denoted by $(M, g, V, \lambda)$, and when $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, the manifold $M$ is called shrinking, steady, or expanding, respectively. If in equation (1.2) the function $\lambda$ is allowed to be a smooth function on $M$, then $M$ is called an almost Ricci soliton.

In the special case where the vector field is of the form $V = \nabla f$, with $f : M \longrightarrow \mathbb{R}$ a smooth function on $M$, equation (1.2) reduces to

$$\mathrm{Ric} + Hess\, f = \lambda g,$$

and in this case $M$ is called a gradient Ricci soliton. Moreover, when $\lambda$ is a smooth function, $M$ is called an almost gradient Ricci soliton.

For further details on Ricci solitons and volume comparison results on such manifolds, see [1, 4].

One of the most beautiful results in volume comparison theory, due to Cheng, states that if a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)H$, where $H$ is a positive constant, has diameter equal to $\dfrac{\pi}{\sqrt{H}}$, then it is isometric to the sphere of radius $\dfrac{1}{\sqrt{H}}$. Moreover, volume comparison theorems have deep connections with problems related to the intrinsic geometry of spaces.

Milnor first conjectured that a complete Riemannian manifold $M$ with $\mathrm{Ric} \geq 0$ has a finitely generated fundamental group, and later proved this conjecture [11]. Furthermore, Gromov and Gallot independently established a beautiful connection between volume comparison and the Betti numbers of a manifold by proving the following result [10, 15]: for a closed Riemannian manifold $M^n$ satisfying

$$\mathrm{Ric} \geq (n-1)H, \qquad diam(M) \leq D,$$

where $diam(M)$ denotes the diameter of the manifold, the first Betti number $b_1(M)$ is uniformly bounded, and moreover, if $HD^2$ is sufficiently small, then $b_1(M) \leq n$.

For further applications of comparison theorems on Riemannian manifolds, see [7, 9, 12]. Recently, Zhu and Zhang [20] obtained interesting results on comparison theorems by imposing conditions on the Ricci curvature. They considered a special type of Ricci curvature of the form

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_V g,$$

where Ric, $V$, and $\mathcal{L}_V$ denote the Ricci curvature, a smooth vector field on the manifold, and the Lie derivative along $V$, respectively. Their results were generalized to the Bakry–Émery Ricci curvature in [17].

In this paper, we first assume a boundedness condition on the Ricci curvature involving a vector field $X$ as follows:

$$\tilde{\mathrm{Ric}} := \mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g \geq (n-1)k, \qquad |X|(y) \leq \frac{L}{d(y,O)^{\alpha}}, \quad \forall y \in M. \tag{1.3}$$

Then, in the special case where the vector field is gradient, we establish the corresponding comparison theorems.


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## 2. Main Results

In this section, by imposing certain conditions on the Ricci curvature involving vector fields in a special case where the vector field is bounded, we prove Laplacian comparison and volume comparison theorems on Riemannian manifolds. Then, using the results of these theorems, we establish the segment inequality for $W \subseteq M^n$.

(Laplacian Comparison) Let $(M^n, g)$ be a Riemannian manifold with a fixed point $O \in M$ such that the following conditions hold:

$$\tilde{Ric} := Ric + \frac{1}{2}\mathcal{L}_V g \geq (n-1)k, \quad |V|(y) \leq \frac{L}{d(y,O)^\alpha}, \tag{2.1}$$

for any $k \leq 0$ and any $y \in M$, where $d(y,O)$ denotes the distance from $O$ to $y$, and $L \geq 0$ and $0 \leq \alpha < 1$ are constants. Let $r = d(x,y)$ and let $\gamma : [0,r] \to M$ be a normal minimal geodesic joining $x$ to $y$. Then

$$\Delta r \leq \frac{(n-1)k}{3} r + \langle V, \nabla r\rangle + \frac{C(\alpha)L}{r^\alpha} + \frac{n-1}{r}, \tag{2.2}$$

where $\Delta$ and $\nabla$ denote the Laplacian and the gradient, respectively, and $C(\alpha)$ is a constant depending only on $\alpha$.

*Proof.* By the Bochner formula, we have

$$0 = \frac{1}{2}\Delta|\nabla r|^2 = |\nabla^2 r|^2 + \langle \nabla\Delta r, \nabla r\rangle + Ric(\nabla r, \nabla r).$$

Using the Cauchy–Schwarz inequality

$$|\nabla^2 r|^2 \geq \frac{(\Delta r)^2}{n-1},$$

we obtain

$$\frac{\partial}{\partial r}\Delta r + \frac{(\Delta r)^2}{n-1} \leq -Ric(\nabla r, \nabla r),$$

which is equivalent to

$$\frac{1}{2}\frac{\partial}{\partial r}(r^2\Delta r) + \frac{1}{n-1}\left(\Delta r - \frac{n-1}{r}\right)^2 \leq \frac{n-1}{r^2} - Ric(\nabla r, \nabla r). \tag{2.3}$$

Multiplying both sides of (2.3) by $r^2$ and integrating from 0 to $r$, we obtain

$$\Delta r \leq \frac{n-1}{r} - \frac{1}{r^2}\int_0^r t^2 Ric(\gamma'(t), \gamma'(t))\, dt. \tag{2.4}$$

At each point $\gamma(t)$, choose an orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ with $e_1 = \gamma'(t)$. Then

$$Ric(\gamma'(t), \gamma'(t)) = Ric(e_1, e_1) \geq (n-1)k - \frac{1}{2}\mathcal{L}_V g(e_1, e_1) = (n-1)k - \partial_t\langle V, \gamma'(t)\rangle.$$

http://dx.doi.org/10.22108/xxxxxxxxxxxxxxxxx

Hence, (2.4) yields

$$\begin{aligned}
\Delta r - \frac{n-1}{r} &\leq \frac{1}{r^2}\int_0^r \left[t^2\frac{\partial}{\partial t}\langle V,\gamma'(t)\rangle + (n-1)kt^2\right]dt \\
&\leq \frac{(n-1)k}{3}r + \frac{1}{r^2}\left[t^2\langle V,\gamma'(t)\rangle\Big|_0^r - 2\int_0^r t\langle V,\gamma'(t)\rangle dt\right] \\
&\leq \frac{(n-1)k}{3}r + \langle V,\nabla r\rangle - \frac{2}{r^2}\int_0^r t\langle V,\gamma'(t)\rangle dt. \qquad (2.5)
\end{aligned}$$

Let $d_0 = d(x,O)$.

- If $r \leq d_0$, then $d(\gamma(t),O) \geq d_0 - t$ and

$$r^{1-\alpha} + (d_0-r)^{1-\alpha} \geq d_0^{1-\alpha}.$$

Thus,

$$\begin{aligned}
-\frac{2}{r^2}\int_0^r t\langle V,\gamma'(t)\rangle dt &\leq \frac{1}{r^2}\int_0^r 2t\frac{L}{(d_0-t)^\alpha}dt \\
&\leq \frac{C(\alpha)L}{r}\left[-(d_0-t)^{1-\alpha}\Big|_0^r\right] \\
&= \frac{C(\alpha)L}{r}\left[d_0^{1-\alpha} - (d_0-r)^{1-\alpha}\right] \\
&\leq \frac{C(\alpha)L}{r^\alpha}, \qquad (2.6)
\end{aligned}$$

where $C(\alpha)$ is a constant depending only on $\alpha$. If we set $x = \frac{r}{d_0-r}$, then

$$x^{1-\alpha} + 1 \geq (1+x)^{1-\alpha}, \quad x > 0.$$

Substituting (2.6) into (2.5), we obtain

$$\Delta r - \frac{n-1}{r} \leq \frac{(n-1)k}{3}r + \langle V,\nabla r\rangle + \frac{C(\alpha)L}{r^\alpha}. \qquad (2.7)$$

- If $r > d_0$, then for any $d_0 \leq t \leq r$ we have $d(\gamma(t),O) \geq t - d_0$.

$$\begin{aligned}
-\frac{2}{r^2}\int_0^r t\langle V,\gamma'(t)\rangle dt &\leq \frac{1}{r^2}\int_0^{d_0} 2t\frac{L}{(d_0-t)^\alpha}dt + \frac{1}{r^2}\int_{d_0}^r 2t\frac{L}{(t-d_0)^\alpha}dt \\
&\leq \frac{C(\alpha)Ld_0}{r^2}d_0^{1-\alpha} + \frac{C(\alpha)L(r-d_0)^{1-\alpha}}{r} \\
&= \frac{C(\alpha)L}{r^\alpha}. \qquad (2.8)
\end{aligned}$$

Therefore, from (2.8) we conclude

$$\Delta r - \frac{n-1}{r} \leq \frac{(n-1)k}{3}r + \langle V,\nabla r\rangle + \frac{C(\alpha)L}{r^\alpha}. \qquad (2.9)$$

The proof is complete. □


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(Volume Comparison) Let $M^n$ be a Riemannian manifold satisfying condition (2.1). Then

$$w(r,\theta) \le e^{C(\alpha)Lr^{1-\alpha}+(n-1)kr^2} r^{n-1},$$

where $w = w(r,\theta)$ is the volume density in geodesic polar coordinates, and it is taken to be zero at the cut locus.

*Proof.* We know that

$$\partial_r w(r,\theta) = w(r,\theta)\Delta r.$$

From (2.9) we obtain

$$\partial_r \ln w(r,\theta) = \frac{\partial_r w}{w} = \Delta r \le -\frac{(n-1)k}{3} r + \frac{n-1}{r} + \langle V, \nabla r\rangle + \frac{C(\alpha)L}{r^\alpha}. \tag{2.10}$$

To continue the proof, we consider the following cases:

- Case 1: $r_1 < r_2 \le d_0$,
- Case 2: $d_0 \le r_1 < r_2$,
- Case 3: $r_1 \le d_0 \le r_2$.

Using (2.1) and integrating inequality (2.10), we obtain

$$\partial_r \ln w(r,\theta) \le \frac{n-1}{r} - \frac{(n-1)k}{3} r + \frac{L}{(d_0-r)^\alpha} + \frac{C(\alpha)L}{r^\alpha}.$$

Integrating both sides from $r_1$ to $r_2$ yields

$$\begin{aligned}
\ln \frac{w(r_2,\theta)}{w(r_1,\theta)} &\le \ln\left(\frac{r_2}{r_1}\right)^{n-1} - \frac{(n-1)k}{6}(r_2^2 - r_1^2) \\
&\quad + C(\alpha)L(r_2^{1-\alpha} - r_1^{1-\alpha}) + C(\alpha)L\left[(d_0-r_1)^{1-\alpha} - (d_0-r_2)^{1-\alpha}\right] \\
&\le \ln\left(\frac{r_2}{r_1}\right)^{n-1} - \frac{(n-1)k}{6} r_2^2 + C(\alpha)Lr_2^{1-\alpha}.
\end{aligned}$$

In particular, for $r_1 \le d_0$, (**??**) becomes

$$\ln \frac{w(d_0,\theta)}{w(r_1,\theta)} \le \ln\left(\frac{d_0}{r_1}\right)^{n-1} + \frac{(n-1)k}{6} d_0^2 + C(\alpha)Ld_0^{1-\alpha}. \tag{2.11}$$

Note that

$$\langle V, \nabla r\rangle \le |V|(\gamma(r)) \le \frac{L}{(r-d_0)^\alpha}.$$

Thus, from (2.10) we obtain

$$\begin{aligned}
\ln \frac{w(r_2,\theta)}{w(r_1,\theta)} &\le \ln\left(\frac{r_2}{r_1}\right)^{n-1} + \frac{(n-1)k}{6}(r_2^2 - r_1^2) \\
&\quad + C(\alpha)L(r_2^{1-\alpha} - r_1^{1-\alpha}) + C(\alpha)L\left[r_2^{1-\alpha} - (r_1-d_0)^{1-\alpha}\right] \\
&\le \ln\left(\frac{r_2}{r_1}\right)^{n-1} + \frac{(n-1)k}{6} r_2^2 + C(\alpha)Lr_2^{1-\alpha}.
\end{aligned}$$

Moreover, for $r_2 \geq d_0$ we have

$$\ln \frac{w(r_2,\theta)}{w(d_0,\theta)} \leq \ln \left(\frac{r_2}{d_0}\right)^{n-1} + \frac{(n-1)k}{6} r_2^2 + C(\alpha) L r_2^{1-\alpha}. \tag{2.12}$$

Substituting (2.11) into (2.12), we obtain

$$\ln \frac{w(r_2,\theta)}{w(r_1,\theta)} \leq \ln \left(\frac{r_2}{r_1}\right)^{n-1} + \frac{(n-1)k}{6} r_2^2 + C(\alpha) L r_2^{1-\alpha}. \tag{2.13}$$

Therefore, from (**??**), (**??**), and (2.13), we conclude

$$\frac{w(r_2,\theta)}{w(r_1,\theta)} \leq e^{C(\alpha) L r_2^{1-\alpha} + (n-1)k r_2^2} \left(\frac{r_2}{r_1}\right)^{n-1}. \tag{2.14}$$

Letting $r_1 \to 0$, we obtain

$$w(r,\theta) \leq e^{C(\alpha) L r^{1-\alpha} + (n-1)k r^2} r^{n-1}, \quad \forall r > 0.$$

□

2.1. **Bakry–Émery Ricci Condition.** In this subsection, we consider the special case in which $V$ is a gradient vector field. In fact, we study the Bakry–Émery Ricci curvature. Let $h$ be a smooth function and set $V = \nabla h$. Assume that

$$Ric + Hess\, h \geq (n-1)k. \tag{2.15}$$

Since, by integration by parts, the condition on the gradient of the potential can be reduced to a single integral, in this case the following condition is imposed on $h$:

$$|h(y) - h(z)| \leq L_2 d(y,z)^{\alpha}, \qquad \sup_{x \in M,\, 0 \leq r \leq 1} \left( r^{\beta} \|\nabla h\|^{*}_{q, B_{(x,r)}} \right) \leq L_3, \tag{2.16}$$

for any $y, z \in M$ with $d(y,z) \leq 1$. Here $L_2, L_3 \geq 0$, $0 < \alpha < 1$, $0 < \beta < 1$, and $q \geq 1$ are constants, and

$$\|\nabla h\|^{*}_{q, B_{(x,r)}} = \left( \fint_{B_{(x,r)}} |\nabla h|^{q}(y)\, dV(y) \right)^{\frac{1}{q}}.$$

In this setting, the non-collapsing condition is no longer required. The above assumption on $h$ indicates that $h$ is uniformly continuous and that $|\nabla h|$ has a singularity of order less than 1. It is easy to see that the local properties of $h$ on balls of radius 1 also hold.

Assume that conditions (2.15) and (2.16) hold.

(1) Let $r = d(x,y)$ be the distance between an arbitrary point $y$ and a fixed point $x$, and let $\gamma : [0,r] \to M$ be a minimal normal geodesic with $\gamma(0) = x$ and $\gamma(r) = y$. Then

$$\Delta r - \frac{n-1}{r} \leq \frac{(n-1)k}{3} r + \frac{4L_2}{r^{1-\alpha}} + \langle \nabla h, \nabla r \rangle, \qquad \forall r < 1, \tag{2.17}$$


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and

$$\frac{\partial}{\partial r}\frac{\omega(r,\theta)}{r^{n-1}} \le \left[\frac{(n-1)k}{3}r + \frac{4L_2}{r^{1-\alpha}} + \langle \nabla h, \nabla r\rangle\right]\frac{\omega(r,\theta)}{r^{n-1}}. \tag{2.18}$$

(2) For any $0 < r_1 < r_2 \le 1$, we have

$$\frac{Vol(B_{(x,r_2)})}{r_2^n} \le e^{(n-1)k(r_2^2-r_1^2)+L_3(r_2-r_1)^{1-\beta}+4L_2(r_2-r_1)^\alpha}\frac{Vol(B_{(x,r_1)})}{r_1^n}. \tag{2.19}$$

The proof can be found in [16, 20].

Similarly, by considering a lower bound for an almost Ricci soliton, the results are obtained as follows:

$$\Delta r - \frac{n-1}{r} \le \frac{(n-1)k}{3}r + \frac{4L_2}{r^{1-\alpha}} + \langle \nabla h, \nabla r\rangle, \qquad \forall r < 1,$$

and

$$\frac{\partial}{\partial r}\frac{\omega(r,\theta)}{r^{n-1}} \le \left[\frac{(n-1)k}{3}r + \frac{4L_2}{r^{1-\alpha}} + \langle \nabla h, \nabla r\rangle\right]\frac{\omega(r,\theta)}{r^{n-1}}.$$

Moreover, for any $0 < r_1 < r_2 \le 1$, we have

$$\frac{Vol(B_{(x,r_2)})}{r_2^n} \le e^{\frac{(n-1)k}{3}(r_2^2-r_1^2)+L_3(r_2-r_1)^{1-\beta}+4L_2(r_2-r_1)^\alpha}\frac{Vol(B_{(x,r_1)})}{r_1^n}.$$

According to the following lemma, when in the inequality $Ric + \frac{1}{2}\mathcal{L}_V g \ge (n-1)k$ we have $V = \nabla h$, assumption (2.16) is in fact weaker than

$$|\nabla h(y)| \le \frac{L}{d(y,O)^\alpha}, \tag{2.20}$$

for any $y \in M$ and a fixed point $O \in M$, with constants $L \ge 0$ and $0 \le \alpha < 1$, together with

$$Vol(B_{(x,r)}) \ge \rho, \tag{2.21}$$

for any $x \in M$ and some fixed $\rho > 0$.

Assume that $Ric + Hess\, h \ge (n-1)k$ and that (2.20) and (2.21) hold. Then for all $q \in \left(n, \frac{n}{\alpha}\right)$,

$$|h(y) - h(z)| \le \frac{2L}{1-\alpha}d(y,z)^{1-\alpha}, \tag{2.22}$$

and for any $y, z \in M$,

$$\sup_{x\in M,\, 0<r\le 1} r^\alpha \|\nabla h\|^*_{q,B_{(x,r)}} \le C(n, L, (n-1)k, \alpha, \rho)\, L. \tag{2.23}$$

For the proof, see [20].

Similarly, by following the above argument for an almost gradient Ricci soliton, we obtain

$$\sup_{x\in M,\, 0<r\le 1} r^\alpha \|\nabla h\|^*_{q,B_{(x,r)}} \le C(n, L, (n-1)k, \alpha, \rho)\, L.$$

The next result describes a special case of gradient Ricci solitons for which stronger results hold. A complete Riemannian manifold $(M, g_{ij})$ is called a gradient Ricci soliton if

$$R_{ij} + \nabla_i \nabla_j h = (n-1)k\, g_{ij}, \tag{2.24}$$

where $(n-1)k$ is a constant and $h$ is a smooth function on $M$. The cases $(n-1)k > 0$, $(n-1)k = 0$, and $(n-1)k < 0$ correspond respectively to shrinking, steady, and expanding solitons. Ricci solitons not only represent self-similar solutions to the Ricci flow, but more importantly, they arise as singularity models; that is, blow-up limits in many Ricci flow situations are Ricci solitons. Therefore, the study of Ricci solitons is essential for the analysis of singularities in the Ricci flow.

For a Ricci soliton (2.24), we have

$$R + |\nabla h|^2 - 2(n-1)k\, h = C_0, \tag{2.25}$$

where $R$ is the scalar curvature of $M$ and $C_0$ is a constant. Hence, for shrinking and expanding solitons, i.e. when $(n-1)k \neq 0$, by adding a constant to $h$ we obtain the normalization

$$R + |\nabla h|^2 - 2(n-1)k\, h = 0. \tag{2.26}$$

For steady solitons, the above equation becomes

$$R + |\nabla h|^2 = C_0. \tag{2.27}$$

If $C_0 = 0$, then by the results of Zhang [19] and Chen [3], for steady solitons we have $R \geq 0$. When $C_0 \neq 0$, equation (2.27) implies $R \equiv 0$, and hence, from

$$\Delta R = \langle \nabla R, \nabla h \rangle + 2|Ric|^2,$$

we obtain $Ric \equiv 0$. Therefore, for steady solitons, one must consider the case $C_0 \neq 0$ in (2.27). We normalize the metric $g$ so that the Ricci soliton satisfies

$$R + |\nabla h|^2 = 1. \tag{2.28}$$

If $(M, g)$ is a shrinking or expanding gradient Ricci soliton, normalization (2.26) holds, and if it is a steady soliton, normalization (2.28) holds. According to the results of Zhang and Chen in [19, 3], we have $R \geq 0$ when $(n-1)k \geq 0$, and $R \geq -C_1(n)$ when $(n-1)k < 0$. Moreover, if we assume $|h| \leq L$, then from (2.26) and (2.28) we obtain

$$|\nabla h| \leq \Lambda(n, (n-1)k, L), \tag{2.29}$$


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where

$$\Lambda(n,(n-1)k,L)=\begin{cases}\sqrt{2(n-1)kL}, & (n-1)k>0,\\ 1, & (n-1)k=0,\\ \sqrt{-2(n-1)kL+C_1(n)}, & (n-1)k<0.\end{cases} \tag{2.30}$$

2.2. **The Gradient Ricci Soliton Case.** Assume that $(M,g_{ij})$ is a gradient Ricci soliton satisfying (2.24). If $(n-1)k\neq 0$, assume (2.26) holds, and if $(n-1)k=0$, assume (2.28) holds. Suppose that on $B_{(O,2\delta)}$, for a fixed point $O\in M$ and radius $\delta$, we have

$$|h|\leq L. \tag{2.31}$$

Let $\Lambda(n,(n-1)k,L)$ be as in (2.30). Then the following statements hold.

(1) Let $r=d(x,y)$ be the distance between any two points $x,y\in B_{(O,\delta)}$, and let $\gamma:[0,r]\to M$ be a minimal normal geodesic with $\gamma(0)=x$ and $\gamma(r)=y$. Then, in the distributional sense,

$$\Delta r-\frac{n-1}{r}\leq\frac{|(n-1)k|}{3}r+2\Lambda(n,(n-1)k,L). \tag{2.32}$$

(2) Let $r$ be as defined in (1), and let $\omega(r,\cdot)$ denote the volume element of the metric $g$ in geodesic polar coordinates. Then for any $0<r_1<r_2<d(x,\partial B_{(O,\delta)})$,

$$\frac{\omega(r_2,\cdot)}{r_2^{n-1}}\leq e^{2\Lambda(n,(n-1)k,L)r_2+|(n-1)k|r_2^2}\frac{\omega(r_1,\cdot)}{r_1^{n-1}}. \tag{2.33}$$

Letting $r_1\to 0$, we obtain

$$\omega(r,\cdot)\leq e^{2\Lambda(n,(n-1)k,L)r+|(n-1)k|r^2}\,r^{n-1}, \tag{2.34}$$

for any $0<r<d(x,\partial B_{(O,\delta)})$, and hence

$$Vol(B_{(x,r)})\leq C(n)e^{2\Lambda(n,(n-1)k,L)r+|(n-1)k|r^2}\,r^{n}. \tag{2.35}$$

(3) For any $x\in B_{(O,\delta)}$ and $0<r_1<r_2<d(x,\partial B_{(O,\delta)})$, we have

$$\frac{Vol(B_{(x,r_2)})}{r_2^n}\leq e^{|(n-1)k|(r_2^2-r_1^2)+2\Lambda(n,(n-1)k,L)(r_2-r_1)}\frac{Vol(B_{(x,r_1)})}{r_1^n}. \tag{2.36}$$

For an almost gradient Ricci soliton, assuming an upper bound $L_1$ for the soliton function $h$, we may take $\Lambda(n,(n-1)k,L)=\sqrt{2L_1L}$. The results then become:

$$\Delta r-\frac{n-1}{r}\leq L_1r+2\sqrt{2L_1L},$$

and for $0<r_1<r_2<d(x,\partial B_{(O,\delta)})$,

$$\frac{\omega(r_2,\cdot)}{r_2^{n-1}}\leq e^{2\sqrt{2L_1L}r_2+L_1r_2^2}\frac{\omega(r_1,\cdot)}{r_1^{n-1}}.$$

Letting $r_1 \to 0$, we have

$$\omega(r,\cdot) \le e^{2\sqrt{2L_1L}r+L_1r^2}\, r^{n-1},$$

and hence

$$Vol(B_{(x,r)}) \le C(n)e^{2\sqrt{2L_1L}r+L_1r^2}\, r^{n}.$$

Finally, for any $0 < r_1 < r_2 < d(x, \partial B_{(O,\delta)})$,

$$\frac{Vol(B_{(x,r_2)})}{r_2^n} \le e^{2\sqrt{2L_1L}(r_2-r_1)+L_1(r_2^2-r_1^2)}\frac{Vol(B_{(x,r_1)})}{r_1^n}.$$

These volume comparison theorems were studied and proved by Cao and Zhou [2] for shrinking gradient Ricci solitons. Moreover, Zhang [18] showed that for this class of gradient Ricci solitons, if both the volume of $M$ and the metric are bounded, then the potential function and the Ricci curvature are also bounded.

As an important application of the volume comparison theorems, analogous to [13], we have the following result.

Let $(M, g)$ be a Riemannian manifold with $\tilde{\mathrm{Ric}} \ge (n-1)K$. Let $f : M \to [0,\infty)$ be a smooth function, and let $A, B \subset W \subset M$. Suppose that for any $x \in A$, $y \in B$, and $t \in [0,1]$, the geodesic segment $c_{x,y}(t) : [0,1] \to M$ joining $x$ and $y$ satisfies $c_{x,y}(t) \in W$, and that $\operatorname{diam} W \le D$. Then

$$\int_{A\times B}\int_0^1 f\circ c_{x,y}(t)\,dt\,Vol_x \wedge Vol_y \le C(\alpha, L, n)\,(Vol\,A + Vol\,B)\int_W f\,Vol. \tag{2.37}$$

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