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RECOGNITION OF THE GROUP $E_6(5)$ BY PRIME GRAPH

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Abstract. If n is an integer, the set of all prime divisors of n is denoted by $\pi(n)$. Let G be a finite group. Then $\pi(|G|)$ is denoted by $\pi(G)$. The prime graph of G, denoted by $\Gamma(G)$, is constructed as follows: the vertex set is $\pi(G)$, and two distinct primes p and q are connected by an edge if and only if G has an element of order pq. A finite nonabelian simple group P is called quasirecognizable by prime graph, if each finite group G with $\Gamma(G) = \Gamma(P)$ has a unique composition factor isomorphic to P. We denote by $k(\Gamma(G))$ the number of isomorphism classes of finite groups H satisfying $\Gamma(G) = \Gamma(H)$. Given a natural number r, a finite group G is called r-recognizable by prime graph if $k(\Gamma(G)) = r$ and if $k(\Gamma(G)) = 1$, is called recognizable . In this paper, as the main result, we show that if G is a finite group such that $\Gamma(G) = \Gamma(E_6(5))$, then $G \cong E_6(5)$.

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Then $\pi(|G|)$ is denoted by $\pi(G)$. We construct the prime graph of G, which is denoted by $\Gamma(G)$, as follows: the vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge if and only if G has an element of order pq. Let m and n be natural numbers. We write $m \sim n$ if and only if for all prime divisors $r \in \pi(m)$ and $s \in \pi(n)$, r is adjacent to s in $\Gamma(G)$. The spectrum of a finite group G, which is denoted by $\pi_e(G)$, is the set of its element orders. Let G be a finite group and $r \in \pi(G)$. We denote by $\rho(G)$ some independent set of vertices in $\Gamma(G)$ with the maximal number of elements. Also some independent set of vertices in $\Gamma(G)$ containing r with the maximal number of elements is

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denoted by $\rho(r, G)$. Put $t(G) = |\rho(G)|$ and $t(r, G) = |\rho(r, G)|$. We denote by $M(G)$, the set of orders of maximal abelian subgroups of G.

A finite nonabelian simple group P is called quasirecognizable by prime graph, if each finite group G with $\Gamma(G) = \Gamma(P)$ has a unique composition factor isomorphic to P. We denote by $k(\Gamma(G))$ the number of isomorphism classes of finite groups H satisfying $\Gamma(G) = \Gamma(H)$. Given a natural number r, a finite group G is called r-recognizable by prime graph if $k(\Gamma(G)) = r$ and unrecognizable if $k(\Gamma(G))$ is infinite. Usually a 1-recognizable group by prime graph is called a recognizable group by prime graph. So far, the recognizability of many finite simple groups by prime graph has been studied. See [7, 8, 9, 10, 11, 12, 13, 14, 18, 19, 26]. Note that if G is recognizable by $\Gamma(G)$, then it is also recognizable by $\pi_e(G)$ and $M(G)$, but the converse is not true in general. In [20], we proved that if q is not a Mersenne prime, then every finite group with the same orders of maximal abelian subgroups as $A_2(q)$, is isomorphic to $A_2(q)$ or an extension of $A_2(q)$ by a subgroup of the outer automorphism group of $A_2(q)$. In [22], we proved that if $L = \mathrm{PSU}_3(q)$, where q is not a Fermat prime, then every finite group with the same set of orders of maximal abelian subgroups as L is an almost simple group with socle $\mathrm{PSU}_3(q)$. In [21], we proved that if G is a finite group such that $M(G) = M(E_6(q))$, then G has a unique nonabelian composition factor which is isomorphic to $E_6(q)$. In [4, 16], it is proved that $E_6(2)$ and ${}^2E_6(2)$, are recoginable by their prime graphs. Also in [15], it is proved that $E_6(3)$ and ${}^2E_6(3)$ are recognizable by their prime graphs. In this paper we prove that $E_6(5)$ is recognizable by its prime graph.

2. Main Results

Lemma 2.1. [24, Theorem 1] *Let G be a finite group with $t(G) \geq 3$ and $t(2, G) \geq 2$. Then the following hold:*

(1) There exists a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$ for a maximal normal solvable subgroup K of G.

(2) For every independent subset ρ of $\pi(G)$ with $|\rho| \geq 3$ at most one prime in ρ divides the product $|K| \cdot |\overline{G}/S|$. In particular, $t(S) \geq t(G) - 1$.

(3) One of the following holds:

(a) Every prime $r \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$ does not divide the product $|K| \cdot |\overline{G}/S|$; in particular, $t(2, S) \geq t(2, G)$.

(b) There exists a prime $r \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$; in which case $t(G) = 3$, $t(2, G) = 2$, and $S \cong Alt_7$ or $A_1(q)$ for some odd q.

Lemma 2.2. [23, Lemma 1] *Suppose that N is a normal elementary abelian subgroup of a finite group G and $H = G/N$. Define an authomorphism $\phi : H \to \mathrm{Aut}(N)$ as follows: $n^{\phi(gN)} = n^g$. Then $\Gamma(G) = \Gamma(N \rtimes_\phi H)$*

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Remark 2.3. *By [2], we know that* $\pi(E_6(5)) = \{2, 3, 5, 7, 11, 13, 19, 31, 71, 313, 601, 829\}$. *Using* [25, Prop 2.5, Prop 3.2, and Prop 4.5], *we get that* $t(E_6(5)) = 5$ *and* $t(2, E_6(5)) = 3$. *Also* $\rho(2, E_6(5)) = \{2, 19, 601, 829\}$.

Theorem 2.4. *If G is a finite group such that $\Gamma(G) = \Gamma(E_6(5))$, then $G \cong E_6(5)$.*

3. Summary of Proofs

By Remark 2.3 and Lemma 2.1, there exists a nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$ for a maximal normal solvable subgroup K of G and $t(S) \geq 4$.

Step 1. We prove that K is nilpotent. We know that G acts on K by conjugation. Since by Remark 2.3, $829 \in \rho(2, G)$, so $829 \notin \pi(K)$ by Lemma 2.1. Suppose that $a \in G$ and $|a| = 829$. By [25, Prop 2.5, Prop 3.2, and Prop 4.5], 829 is nonadjacent to all vertices of $\pi(K)$. Therefore the action of a on K is fixed-point free. Hence by Thompson's theorem, K is nilpotent.

Step 2. We prove that $S \cong E_6(5)$. By Remark 2.3, 829 is the largest prime in $\pi(S)$. Using [27], S can be a group from the following groups:

$$L_3(5^3), L_4(5^3), L_2(829^2), S_4(829), U_3(829), E_6(5), U_4(829)$$

. We prove that $S \cong E_6(5)$.

Step 3. We prove that $G/K \cong E_6(5)$. By [2], $|\mathrm{Out}(E_6(5))| = 2$, so either $G/K \cong E_6(5)$ or $G/K \cong \mathrm{Aut}(E_6(5))$. Using graph automorphism of order 2, we conclude that, $G/K \cong E_6(5)$.

Step 4. We show that $\pi(K) \subseteq \{5, 31\}$. Assume that $p \in \pi(K)$. Since K is nilpotent, by [13] K is an elementary abelian p-group.We show that $601 \sim p$ in $\Gamma(G)$. So by [25], $p \in \{5, 31\}$.

Step 5. We prove that $\pi(K) \subseteq \{5\}$. Suppose that $31 \in \pi(K)$. Using [17], we see that $O_8(5) < E_6(5)$. According to the orders of the groups $O_8(5)$ and $E_6(5)$, we get that their Sylow 13-subgroups are isomorphic. By [6], we see that $E_6(5)$ has a torus that is a direct product of two cyclic groups of order $(5-1)(5^2+1)$. Since 13 divides into 5^2+1, hence we conclude that the Sylow 13-subgroup is non-cyclic. Now consider a Sylow 13-subgroup P of $E_6(5)$. Denote by $\widetilde{P}$ the full preimage of P in G. The conjugation action of 13-elements of $\widetilde{P}$ on K is fixed-point free, so $\widetilde{P}$ is a Frobenius group. By [3], P is cyclic, which is a contradiction.

Step 6. We prove that $K = 1$. Suppose that $K \neq 1$. Similarly to the above, we can assume that K is elementary abelian. By [17], $F_4(5) \leq G/K$. Consider the action of $F_4(5)$ on K by Lemma 2.2. According to [5], any element of order 601 in $F_4(5)$ fixes some non-identity element in K. Thus $601 \sim 5$ in $\Gamma(G)$, which is a contradiction. So $K = 1$. Therefore $G \cong E_6(5)$ and the proof of the Main Theorem is completed.

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