https://math-sci.ui.ac.ir

Mathematics and Society

ISSN (print): 2345-6493, ISSN (on-line): 2345-6507

Vol. 11 No. 2 (2026), pp. 81–96.

https://www.ui.ac.ir

QUANTUM LOGIC

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Abstract. This article employs a descriptive-analytical method with an algebraic and axiomatic approach to address quantum logic. Through several thought experiments, it is demonstrated that the distributive law for propositions does not hold in quantum mechanics. Therefore, the logic governing this domain of science is not classical logic. In fact, the algebra corresponding to the logic of quantum propositions is an orthomodular lattice, which is why the distributive law is not necessarily valid for such propositions. In quantum logic, various definitions are offered for the implication operator. Since none of these operators satisfy the deduction theorem, a suitable implication operator with this property must be defined. This objective is achieved by defining a unified quantum logic. By constructing a Lindenbaum-Tarski algebra for this unified quantum logic, it can be shown that it is indeed a quantum logic. Dishkant presented an embedding of quantum logic into the Br-modal logic system. This provides an appropriate semantics for quantum logic. A suitable model for this logic is the closed subspaces of a Hilbert space.

1. Introduction

Quantum mechanics describes the behavior of matter and energy at atomic and subatomic scales, where classical intuition fails. Early physics, following Newton, treated light as particles, but later discoveries revealed wave-like behavior. By the early 20th century, experiments such as the photoelectric effect suggested that light sometimes behaves like particles. Similarly, electrons, initially assumed to behave strictly

Keywords: Quantum logic, modal logic, Hilbert space.

Article Type: Promotional Paper.

Communicated by Rasoul Roknizadeh.

Received: 02-12-2024, Accepted: 30-09-2025, Published Online: 02-11-2025.

Cite this article: A. Moazzen, Quantum logic, *Mathematics and Society*, **11** no. 2 (2026) 81–96. https://dx.doi.org/10.22108/msci.2025.129248.1712 .

as particles, were found to exhibit wave-like properties. This duality produced growing conceptual ambiguities that were resolved between 1926–1927 through the foundational work of Schrödinger, Heisenberg, and Born.

The first formalism of quantum mechanics, matrix mechanics, was introduced by Werner Heisenberg in a seminal paper. In this framework, physical quantities such as momentum and kinetic energy are represented by infinite matrices, whose eigenvalues correspond to possible measurable values. The non-commutativity of matrix multiplication implies that some observables cannot be simultaneously measured with arbitrary precision—this underlies Heisenberg's uncertainty principle. Born and Jordan formulated an explicit version of the uncertainty relation: if P and Q are matrices representing position and momentum, then:

$$PQ - QP = \frac{h}{\pi i}.$$

In 1926, Erwin Schrödinger published his partial differential equation—the Schrödinger equation—which became the most famous equation in physics. Schrödinger proved that its solutions in L^2 space correspond to experimentally observed electron energy levels in hydrogen. The stationary-state wavefunction is written $\psi(x, y, z)$, and in non-stationary cases includes time t. Schrödinger himself did not initially provide a physical interpretation of ψ, but Max Born proposed the probabilistic interpretation: $|\psi|^2 dxdydz$ gives the probability of locating a particle in the infinitesimal volume element $dxdydz$. This probabilistic interpretation is now standard.

1.1. **Research Background in Quantum Logic.** The reconsideration of deductive logic to align with quantum theory began in the early 20th century. Birkhoff and von Neumann first described quantum logic using non-classical lattices. Later, researchers like Dalla Chiara and Giuntini analyzed semantic and epistemological aspects, showing that quantum logic carries implications beyond pure mathematics [4]. Cohen attempted to model quantum propositions using Hilbert space subspaces [3], while Rédei analyzed the philosophical gap between classical and quantum logic [11]. In Persian scholarship, Ebadi critically examined Putnam's interpretation of quantum logic and highlighted distinctions between logical systems [5].

1.2. **Critique of Classical Logic and Need for Quantum Logic.** Classical logic has been the standard analytical framework for scientific theories, relying on principles like the law of excluded middle and causality. However, quantum phenomena such as superposition and entanglement reveal the inadequacy of these principles at the microscopic level. Logical analysis shows that the lattice structure of propositions about quantum systems is not Boolean but orthomodular and non-distributive. This necessitates a new logic compatible with quantum features. Persisting with classical logic leads to distorted or incomplete interpretations of quantum phenomena [11]. Hence, quantum logic emerges as an essential framework.

https://dx.doi.org/10.22108/msci.2025.129248.1712

1.3. **Axioms of Quantum Mechanics.** A key difference between classical and quantum mechanics is that in the classical view, a quantity A of a system S has a definite value independent of measurement, and measuring does not affect the state. In quantum mechanics, measurements yield different possible values with probabilities depending on the system's state, and the act of measurement generally alters the system. The formal axioms are:

(1) **State representation:** There is a one-to-one correspondence between physical states of S and rays (one-dimensional subspaces) of a Hilbert space H.
(2) **Observables:** Physical quantities correspond to Hermitian operators on H.
(3) **Measurement outcomes:** Possible measurement results are eigenvalues of the corresponding Hermitian operator. Degenerate cases occur when an eigenspace has dimension greater than one.
(4) **Probabilities:** If the system is in normalized state x and we measure observable A with eigenvector y_n for eigenvalue a_n, the probability of obtaining a_n is:

$$p(a_n) = |\langle y_n, x\rangle|^2.$$

(5) **State collapse:** Upon measurement yielding a, the state collapses to the normalized projection onto the eigenspace corresponding to a. The combination of two systems is represented by the tensor product of their Hilbert spaces [6].

1.4. **Wave–Particle Duality and the Double-Slit Experiment.** To understand electron behavior, it is instructive to compare particles (bullets) and waves (water).

Bullets: Imagine firing bullets at a barrier with two narrow slits. The bullets pass through individual slits and strike a detection wall. The probability of hitting a point x is additive:

$$p_{12}(x) = p_1(x) + p_2(x).$$

Water Waves: Repeating the experiment with water waves creates interference. The intensity from two slits is:

$$I_{12} = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\theta,$$

where θ is the phase difference. Here, $I_{12} \neq I_1+I_2$, demonstrating constructive and destructive interference.

Electrons: Using electrons instead of bullets or water produces an interference pattern similar to waves, even when electrons are sent one at a time. Initially, one might assert proposition E: "Each electron passes through either slit (1) or slit (2)." If E were true, we would expect:

$$p_{12}(x) = p_1(x) + p_2(x),$$

but experiments show $p_{12} \neq p_1 + p_2$. Thus, E fails: electrons do not behave as classical particles choosing a definite slit. When a strong light source is added to observe the slits, scattered photons reveal which slit the electron uses, and the interference pattern disappears. Measurement collapses the wavefunction, making E effectively true under observation but invalid when unobserved. This demonstrates the central role of measurement in quantum theory and highlights non-classical logical structures.

1.5. **Philosophical and Logical Implications.** This behavior suggests that quantum propositions cannot always be treated as having definite truth values independent of observation. The measurement context determines outcomes, undermining the classical law of excluded middle. The orthomodular lattice structure of quantum propositions matches experimental results, while classical Boolean logic fails. Quantum logic does not replace classical logic universally but extends it to account for phenomena like superposition and entanglement. Scholars like Birkhoff, von Neumann, Cohen, and Rédei argue that this shift has epistemological consequences beyond physics, affecting how knowledge and reasoning are conceptualized [1, 3, 11, 5].

2. Main Results

This section summarizes the main results about quantum logic (QL) presented in the original text. Quantum logic formalizes propositions about quantum systems, departing from classical Boolean logic. Propositions p_j are defined inductively: any propositional variable is a formula, negation $\sim A$ and conjunction $A \wedge B$ of formulas are formulas, and disjunction is defined via De Morgan as $A \vee B =\sim (\sim A \wedge \sim B)$. Multiple implication operators have been proposed to capture the non-classical inference structure of quantum systems—such as $\rightarrow_1$ (Sasaki, Mittelstedt) and $\rightarrow_2$ (Dishkant)—but none satisfies the classical Deduction Theorem.

$$
\begin{aligned}
&A \rightarrow_0 B \doteq \sim A \vee B \\
&A \rightarrow_1 B \doteq \sim A \vee (A \wedge B) && (Sasaki, Mittelstedt) \\
&A \rightarrow_2 B \doteq B \vee (\sim A \wedge \sim B) && (Dishkant) \\
&A \rightarrow_3 B \doteq (A \wedge B) \vee (\sim A \wedge B) \vee (\sim A \wedge \sim B) \\
&A \rightarrow_4 B \doteq (A \wedge B) \vee (\sim A \vee B) \vee ((\sim A \vee B) \wedge B) \\
&A \rightarrow_5 B \doteq (\sim A \wedge \sim B) \vee (\sim A \wedge B) \vee ((\sim A \wedge B) \wedge A
\end{aligned}
$$

Pavicic [9] showed that none of these has priority over another, leading to the Unified Quantum Logic (UQL), which preserves desirable properties like orthomodularity and transitivity ([10]).

The axiomatization of QL uses postulates such as $A \leftrightarrow A$, double negation $A \leftrightarrow \sim\sim A$, and $(A \wedge B) \rightarrow A$. Inference rules include transitivity (R_1), contraposition (R_2), and conjunction introduction (R_3). Dropping

https://dx.doi.org/10.22108/msci.2025.129248.1712

rule R_4 yields the minimal quantum logic MQL.

$$R_1: \quad A \to B \& B \to C \Rightarrow A \to C$$
$$R_2: \quad A \to B \Rightarrow \sim B \to \sim A$$
$$R_3: \quad A \to B \& A \to C \Rightarrow A \to (B \wedge C)$$
$$R_4: \quad A \to B \& (\sim A \wedge B) \to (C \wedge \sim C) \Rightarrow B \to A$$

Adding orthomodularity,

$$OML : A \wedge (\sim A \vee (A \wedge B)) \to B,$$

makes MQL equivalent to full QL. Classical logic CL is recovered from MQL by imposing distributivity:

$$D : (A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C)).$$

Quantum logic is algebraically represented by orthomodular lattices: complemented lattices $(L, \vee, \wedge, ', 0, 1)$ where for each $a \in L$ there exists a' such that $a \vee a' = 1$ and $a \wedge a' = 0$. An ortholattice satisfying $a \leq c \Rightarrow a \vee (a' \wedge c) = c$ is orthomodular ([1]). This lattice-theoretic approach captures the partial ordering and compatibility structure of quantum propositions.

Dishkant embedded quantum logic into modal logic Br^-, providing a semantic interpretation via necessity L and possibility M, linked by $Lp =\sim M \sim p$ ([8]). The modal systems T, S_4, and S_5 are progressively stronger: T includes necessity $Lp \supset p$; adding $Lp \supset LLp$ gives S_4, and adding $Mp \supset LMp$ gives S_5. Brauer's system Br arises by adding $p \supset LMp$ to T, while Br^+ strengthens T with an additional inference rule. These embeddings show quantum logics can be studied within well-developed modal frameworks, connecting operator-algebraic semantics to modal reasoning.

Von Neumann and Birkhoff connected QL to Hilbert space theory: propositions correspond to closed subspaces (or projection operators) of a Hilbert space H ([2]). For propositions p, q with subspaces M_p, M_q, conjunction corresponds to intersection $M_{p\wedge q} = M_p \cap M_q$, disjunction corresponds to the closed linear span $M_{p\vee q} = M_p \oplus M_q$, and negation corresponds to orthogonal complement $M_{\sim p} = M_p^{\perp}$. Logical implication $p \to q$ is identified with subspace inclusion $M_p \subseteq M_q$. This captures non-classical behavior: a vector x can belong to $M_{p\vee q}$ without lying in M_p or M_q individually, reflecting that disjunction in QL may be true even if neither disjunct is individually true.

The compatibility of propositions is formalized: p and q are compatible if there exist mutually orthogonal u, v, w with $p = u \vee v$ and $q = v \vee w$. Classical logic is the special case where all propositions are pairwise compatible; a genuinely quantum logic requires at least one pair of incompatible propositions ([13]; [12]).

Pavicic's unified approach ([10]) shows that by using a uniform implication operator, one can construct an axiomatic system whose Lindenbaum–Tarski algebra is an orthomodular lattice, preserving essential properties such as the orthomodular law, transitivity, and the law of excluded middle. Dishkant's embedding $UQL \hookrightarrow Br^{-}$ demonstrates that QL can be represented within modal logic, ensuring semantic completeness and offering tools from modal semantics.

3. Conclusions

Overall, the results consolidate several approaches: algebraic (orthomodular lattices), modal (via T, S_4, S_5, Br^{-}), and Hilbert-space-based (Birkhoff and von Neumann's construction). These show that quantum logic is weaker than classical logic (due to failure of distributivity), but by adding distributivity or other axioms, classical logic is recovered. The compatibility concept explains why quantum propositions can behave non-classically: two observables may not be simultaneously measurable, corresponding to non-compatible subspaces. Thus, quantum logic formalizes the departure from Boolean reasoning required by quantum mechanics.

The connection between subspace lattices and propositions, first articulated by Birkhoff and von Neumann (1936), remains central: the closed subspaces of a Hilbert space form an orthomodular lattice, and their meet, join, and complement correspond directly to quantum logical operations. Subsequent authors (Hughes & Cresswell, 1996; Pavicic, 1998) have refined the axioms, implication operators, and modal embeddings to provide a unified and semantically rich account of QL. These developments underpin modern approaches in quantum foundations, quantum computation, and categorical quantum mechanics, where logical structure plays a critical role.

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