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G-AMENABILITY FOR DIRECT SUM, TENSOR PRODUCT AND FREE PRODUCT OF VON-NEUMANN ALGEBRAS

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ABSTRACT. For a family of W^*-dynamical systems $(M_i, G, \alpha_i)_{i\in I}$, where G is a locally compact group, we prove that if the direct sum $\bigoplus_{i\in I} M_i$ is G-amenable, then each M_i is also G-amenable. Conversely, if all M_i's form a countable family of G-amenable von-Neumann algebras, then $\bigoplus_{i\in I} M_i$ is G-amenable as well. For two W^*-dynamical systems (M, G, α) and (N, K, β), we show that the von-Neumann tensor product $M\bar{\otimes}N$ is $G\times K$-amenable if and only if M is G-amenable and N is K-amenable. We show that if the group G is inner amenable, then the group von-Neumann algebra $VN(G)$ is also G-amenable. Furthermore, we prove that $VN(G)\bar{\otimes}M$ is G-amenable whenever the action α is inner amenable and M is G-amenable. Finally, we show that von-Neumann algebras M and N are G-amenable if and only if their free product $M\bar{*}N$ is G-amenable. We also prove that the amenability of the group G is equivalent to the G-amenability of $L^\infty(G)\bar{*}L^\infty(G)$.

1. Introduction

Let G be a locally compact group and let $L^\infty(G)$ denote the von-Neumann algebra of essentially bounded measurable functions on G, as introduced in [5]. The concept of amenability for the group G can be described by the existence of a positive, bounded linear functional P of norm one on $L^\infty(G)$ such that for every $s \in G$ and every $f \in L^\infty(G)$, we have

$$P(\ell_s f) = P(f),$$

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where for almost every $t \in G$, the left translation is given by $\ell_s f(t) = f(s^{-1}t)$. For further studies on amenable groups, see [3, 8, 9].

The homomorphism $\ell : G \to \mathrm{Aut}(L^\infty(G))$ defined by $\ell_s f(t) = f(s^{-1}t)$ describes the left translation action of G on the von-Neumann algebra $L^\infty(G)$. The triple $(L^\infty(G), G, \ell)$ thus forms a W^*-dynamical system. It follows that the amenability of the group G is equivalent to the existence of a positive, norm-one, bounded linear functional P on the von-Neumann algebra $L^\infty(G)$ that is invariant under the action of G.

Motivated by this characterization, in 2006 Lau and Paterson, in [6], considered a group action of G on a von-Neumann algebra M forming a W^*-dynamical system, and introduced the notion of G-amenability for the algebra M.

We recall that by a *von-Neumann algebra* M we mean a unital $*$-subalgebra of the algebra of bounded linear operators on a Hilbert space H, which is closed in the strong operator topology.

Moreover, the algebra of all bounded linear operators on H is denoted by $B(H)$, that is,

$$B(H) = \{T \mid T : H \to H \text{ is linear and bounded}\}.$$

A triple (M, G, α) is called a W^*-dynamical system if $\alpha : G \to \mathrm{Aut}(M)$ is a group homomorphism from G into the group of $*$-automorphisms of M, such that for every $m \in M$ and $\xi, \eta \in H$, the map

$$s \mapsto \langle \alpha_s(m)(\xi), \eta \rangle$$

is continuous. Equivalently, due to the boundedness of the set $\{\alpha_s(m) \mid s \in G\}$, the map

$$s \mapsto s \cdot m := \alpha_s(m) \quad (G \to (M, \sigma(M, M_*)))$$

is continuous, where M_* denotes the predual of M and $\sigma(M, M_*)$ is the weak* topology on M.

Similarly, M is called a right G-module if the action of G on M is given by

$$s \mapsto m \cdot s := \alpha_{s^{-1}}(m).$$

According to [11, p. 239], the dual space M^* is a left (right) Banach G-module; that is, the map $s \mapsto s \cdot P$ (respectively, $s \mapsto P \cdot s$) is norm continuous for every $P \in M^*$ and $s \in G$.

For every $s \in G$ and $P \in M^*$, we define:

$$s \cdot P := (\alpha_{s^{-1}})^* (P), \qquad P \cdot s := (\alpha_s)^* (P),$$

where $(\alpha_s)^* : M^* \to M^*$ denotes the adjoint map of $\alpha_s : M \to M$.

Therefore, for all $P \in M^*$, $m \in M$, and $s \in G$, we have:

$$\langle s \cdot P, m \rangle = \langle P, m \cdot s \rangle, \qquad \langle P \cdot s, m \rangle = \langle P, s \cdot m \rangle.$$

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In this paper, by a *state* we mean a positive linear functional of norm one on the von-Neumann algebra M, and the set of all states on M is denoted by $S(M)$; that is,

$$S(M) = \{P \in M^* \mid \|P\| = P(1) = 1\}.$$

For more on group modules, see [10].

As a fundamental example, the triple $(L^\infty(G), G, \ell)$ forms a W^*-dynamical system. It follows that G is amenable if and only if there exists a state $P \in S(L^\infty(G))$ such that for all $f \in L^\infty(G)$,

$$P(\ell_s f) = P(f),$$

which is equivalent to $s \cdot P = P$ for every $s \in G$.

This motivates the following definition in the general setting of W^*-dynamical systems, as originally introduced in [6].

Definition 1.1. *Let (M, G, α) be a W^*-dynamical system. The von-Neumann algebra M is said to be G-amenable if there exists a state $P \in S(M)$ such that $s \cdot P = P$ for all $s \in G$. In this case, P is called a G-invariant state.*

The main goal of this paper is to investigate group amenability for three classes of W^*-dynamical systems. In Section 2, we first consider a family of W^*-dynamical systems $(M_i, G, \alpha_i)_{i\in I}$ along with a locally compact group G. We show that there exists an action of G on the direct sum $\bigoplus_{i\in I} M_i$ such that it forms a W^*-dynamical system. Moreover, we prove that if $\bigoplus_{i\in I} M_i$ is G-amenable, then each M_i is G-amenable as well. Conversely, if all M_i's form a countable family of G-amenable von-Neumann algebras, then the direct sum $\bigoplus_{i\in I} M_i$ is also G-amenable. As a consequence, we obtain a necessary and sufficient condition for the amenability of the group G.

Next, for two W^*-dynamical systems (M, G, α) and (N, K, β), we show that there exists an action of the group $G \times K$ on the von-Neumann tensor product $M\bar{\otimes}N$ such that $M\bar{\otimes}N$ is $G \times K$-amenable if and only if M is G-amenable and N is K-amenable.

Furthermore, we provide a sufficient condition for the G-amenability of the group von-Neumann algebra $VN(G)$. In particular, we prove that if the group G is inner amenable, then $VN(G)$ is G-amenable. We also show that $VN(G)\bar{\otimes}M$ is G-amenable whenever the action α is inner amenable and M is G-amenable.

In Section 3, we consider two W^*-dynamical systems (M_1, G, α_1) and (M_2, G, α_2), and show that an action of G on the free product von-Neumann algebra $M_1\bar{*}M_2$ can be defined. We prove that $M_1\bar{*}M_2$ is G-amenable if and only if both M_1 and M_2 are G-amenable.

Finally, we demonstrate that the amenability of G is equivalent to the G-amenability of the free product $L^\infty(G)\bar{*}L^\infty(G)$.

2. Main Results

Theorem 2.1. *Let $\{(M_i, G, \alpha_i)\}_{i\in I}$ be a family of W^*-dynamical systems. Then:*

(i). *If $\bigoplus_{i\in I} M_i$ is G-amenable, then each M_i is G-amenable.*

(ii). *If I is a countable set and each M_i is G-amenable, then the direct sum $\bigoplus_{i\in I} M_i$ is also G-amenable.*

Corollary 2.2. *Let G be a locally compact group. Then G is amenable if and only if for every countable index set I, the von-Neumann algebra $\bigoplus_{i\in I} L^\infty(G)$ is G-amenable.*

Corollary 2.3. *Let (M, G, α) be a W^*-dynamical system such that G is a countable discrete group. Then M is G-amenable if and only if $\ell^\infty(G, M)$ is G-amenable.*

Suppose that M and N are von-Neumann algebras on Hilbert spaces H_M and H_N, respectively. The von-Neumann algebra generated by the set $\{m \otimes n \mid m \in M, n \in N\}$ (on the tensor product Hilbert space $H_M \otimes H_N$) is called the tensor product of M and N, and is denoted by $M\bar{\otimes}N$; for more details, see [4].

Let (M, G, α) and (N, K, β) be two W^*-dynamical systems. Consider the homomorphism $\alpha \times \beta : G \times K \to \mathrm{Aut}(M\bar{\otimes}N)$ defined by

$$(\alpha \times \beta)_{(s,t)}(m \otimes n) = \alpha_s(m) \otimes \beta_t(n).$$

Then the triple $(M\bar{\otimes}N, G \times K, \alpha \times \beta)$ is a W^*-dynamical system.

Theorem 2.4. *Let (M, G, α) and (N, K, β) be two W^*-dynamical systems. Then, with respect to the W^*-dynamical system $(M\bar{\otimes}N, G \times K, \alpha \times \beta)$, the von-Neumann algebra $M\bar{\otimes}N$ is $G \times K$-amenable if and only if M is G-amenable and N is K-amenable.*

Corollary 2.5. *Let G be a locally compact group. Then G is amenable if and only if the von-Neumann algebra $L^\infty(G \times G)$ is G-amenable in the W^*-dynamical system $(L^\infty(G \times G), G, \ell \times \ell)$.*

Proposition 2.6. *Let G be a locally compact group. If G is inner amenable, then the group von-Neumann algebra $VN(G)$ is G-amenable.*

Corollary 2.7. *Let G be a locally compact group. If G is amenable, then the group von-Neumann algebra $VN(G)$ is G-amenable.*

Corollary 2.8. *If G is a finite or abelian group, then the direct sum of any number of copies of $VN(G)$ is a G-amenable von-Neumann algebra.*

Theorem 2.9. *Let (M, G, α) be a W^*-dynamical system and suppose that the action α is inner amenable. If M is G-amenable, then the tensor product $VN(G)\bar{\otimes}M$ is also G-amenable.*

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Suppose that (M_1, G, α_1) and (M_2, G, α_2) are two W^*-dynamical systems. The free product of the von-Neumann algebras M_1 and M_2, denoted by $M_1 \bar{*} M_2$. Define the map $\alpha_1 \bullet \alpha_2 : G \to \mathrm{Aut}(M_1 \bar{*} M_2)$ by the rule:

$$(\alpha_1 \bullet \alpha_2)(s)(m_{i_1} m_{i_2} \cdots m_{i_n}) = \alpha_{i_1}(s)(m_{i_1})\alpha_{i_2}(s)(m_{i_2}) \cdots \alpha_{i_n}(s)(m_{i_n}),$$

where $m_{i_j} \in M_{i_j}$ and no two consecutive elements belong to the same von-Neumann algebra.

From now on, for simplicity of notation, we write:

$$s \cdot (m_{i_1} m_{i_2} \cdots m_{i_n}) := (\alpha_1 \bullet \alpha_2)(s)(m_{i_1} m_{i_2} \cdots m_{i_n}) = s \cdot m_{i_1}\, s \cdot m_{i_2} \cdots s \cdot m_{i_n}.$$

Lemma 2.10. *Suppose that (M_1, G, α_1) and (M_2, G, α_2) are two W^*-dynamical systems. Then $(M_1 \bar{*} M_2, G, \alpha_1 \bullet \alpha_2)$ is a W^*-dynamical system.*

Theorem 2.11. *Suppose that (M_1, G, α_1) and (M_2, G, α_2) are two W^*-dynamical systems. Then $M_1 \bar{*} M_2$ is G-amenable if and only if both M_1 and M_2 are G-amenable.*

Corollary 2.12. *Suppose that G is a discrete group. Then both $VN(G)$ and $VN(G * G)$ are G-amenable von-Neumann algebras.*

Corollary 2.13. *Suppose that G is a locally compact group. Then G is amenable if and only if $L^\infty(G) \bar{*} L^\infty(G)$ in the W^*-dynamical system $(L^\infty(G) \bar{*} L^\infty(G), G, \ell \bullet \ell)$ is a G-amenable von-Neumann algebra*

3. Conclusions

In this paper, we study the notion of G-amenability for a W^*-dynamical system (M, G, α). By using the direct sum, tensor product, and free product of von-Neumann algebras, we construct new W^*-dynamical systems. We investigate the conditions under which these three classes of von-Neumann algebras are G-amenable, leading to new characterizations for the amenability of the group G. In particular, for two W^*-dynamical systems (M, G, α) and (N, G, β), we conclude that the amenability of G is equivalent to the G-amenability of each of the following von-Neumann algebras:

- the direct sum $M \oplus N$,
- the tensor product $M \bar{\otimes} N$,
- the free product $M \bar{*} N$.

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