https://math-sci.ui.ac.ir

**Mathematics and Society**
ISSN (print): 2345-6493, ISSN (on-line): 2345-6507
Vol. 11 No. 2 (2026), pp. 41–64.


https://www.ui.ac.ir

# SOME RESULTS OF THE $\phi$-BEST PROXIMITY POINT IN THE COMPLETE METRIC SPACE

MARYAM SHAMS* AND FARAH RASHIDI

Abstract. In this paper, we introduce two types of proximal contractions. First, we define the $(\phi, \varphi, \psi, H)$-proximal contraction and subsequently the weak $(\phi, \varphi, \psi, H)$-proximal contraction. Then, we investigate the existence and uniqueness of the $\varphi$-best proximity point for these contractions in a complete metric space, considering specific conditions. One of these specific conditions, required in both theorems, is that the function $\psi$ is non-decreasing and the function $\varphi$ is lower semi-continuous. The main theorems obtained are generalizations and extensions of existing $\varphi$-best proximity point theorems for proximal contractions related to the control function $H$. If, in the main theorems, the two subsets $A$ and $B$ are equal, then the existence and uniqueness of a fixed point for the corresponding self-mappings are obtained. Subsequently, we illustrate the importance and applicability of the main theorems with the help of examples.

## 1. Introduction

The analysis and solution of fixed-point problems in metric spaces have long been a focal point for researchers in various fields of mathematics and applied sciences. The Banach contraction principle, as a fundamental result in this area, plays a significant role in solving differential equations, integral equations, and optimization problems. However, the limitations of this principle have driven researchers to

Keywords: Best proximity point, $(\phi, \varphi, \psi, H)$-proximal contraction, $(\phi, \varphi, \psi, H)$-weak proximal contraction, metric space.

Article Type: Resaerch Paper.

Communicated by Alireza Amini Harandi.

*Corresponding author.

Received: 19-10-2024, Accepted: 12-07-2025, Published Online: 03-11-2025.

**Cite this article:** M. Shams and F. Rashidi, Some results of the $\phi$-best proximity point in the complete metric space, *Mathematics and Society*, **11** no. 2 (2026) 41–64. http://dx.doi.org/10.22108/msci.2025.143009.1706 .

develop and generalize contraction concepts. Wardowski, with the introduction of the $F$-contraction concept [17], took an important step towards generalizing the Banach contraction principle. This concept has inspired extensive research in the field of fixed-point theory, including new concepts of contractions such as $F$-weak contractions [18] and $F$-contractive mappings of Hardy-Rogers-type [3], and has yielded significant results regarding the existence and uniqueness of fixed points. Furthermore, the generalization of these concepts to extended metric spaces has provided new research areas. Meanwhile, Jleli and Samet [10], with the introduction of $\Theta$-contractions and a different approach, obtained new results on fixed points. Shams et al. [14], by introducing $(\psi, a, \varphi, F)$-generalized contraction mappings, investigated the existence conditions for a unique $\varphi$-fixed point. The concept of best proximity point, as a generalization of the concept of fixed point, has recently attracted much attention from researchers. Many researchers have examined the existence and convergence conditions of best proximity points for various mappings [1, 2, 4, 13, 15, 16, 19]. Despite the valuable results available in the field of best proximity point theory and the application of proximal contractions [8, 7], further research is needed to investigate the conditions for the existence and uniqueness of best proximity points in broader classes of proximal mappings. In this article, we address this issue and introduce a new concept of proximal mappings called $(H, \psi, \varphi, \phi)$-proximal contraction and $(H, \psi, \varphi, \phi)$-weak proximal contraction. Then, we prove theorems regarding the existence and uniqueness of a $\varphi$-best proximity point for these mappings and demonstrate the effectiveness of our results by providing examples.

Let $(X, d)$ is a metric space and $T : X \to X$ be a self-map. The set of all fixed points of the map $T$ denoted by $F_T = \{x \in X \mid Tx = x\}$.

Moreover, we show by $\Delta$ the family of all functions $\psi : [0, \infty) \to [0, \infty)$ that satisfy the following conditions.

- $(\psi_1)$ $\psi$ is a non-decreasing function.
- $(\psi_2)$ for every positive sequence $\{\alpha_n\}$ we have $\lim_{n\to\infty} \psi(\alpha_n) = 0$ if and only if $\lim_{n\to\infty} \alpha_n = 0$.
- $(\psi_3)$ is a continuous function.

We also consider $\Theta$ to be a family of all functions $\phi : [0, \infty) \to [0, \infty)$ that satisfy the following conditions:

- $(\phi_1)$ $\phi$ is a non-decreasing function
- $(\phi_2)$ for all $\alpha > 0$ we have $\lim_{n\to\infty} \phi^n(\alpha) = 0$, where $\phi^n$ denotes the $n$th iterate of $\phi$.

**Lemma 1.1.** [11] *If $\phi \in \Theta$, then for all $t > 0$ we have $\phi(t) < t$.*

Liu et al. [11], in 2016, using the functions defined above, first introduced the concept of $(\psi, \phi)$-contraction, and then showed that if a mapping $T : X \to X$ is a $(\psi, \phi)$-contraction, then $T$ has a unique fixed point.

In 2020, Proinov proved the following theorem in a complete metric space [12].


http://dx.doi.org/10.22108/msci.2025.143009.1706

**Theorem 1.2.** [12] *Let $(X,d)$ be a complete metric space and $T : X \to X$ be a mapping such that for every $x, y \in X$ we have:*

$$d(Tx,Ty) > 0 \Rightarrow \psi(d(Tx,Ty)) \le \phi(d(x,y)),$$

*where $\psi, \phi : [0,\infty) \to \mathbb{R}$ satisfy in the following conditions:*

(1) *$\psi$ is non-increasing*
(2) *for all $\alpha > 0$, we have $\phi(\alpha) < \psi(\alpha)$.*
(3) *for all $r > 0$ we have $\limsup_{\alpha\to r^+} \phi(\alpha) < \psi(r^+)$.*

*We denote by $\Psi$ and $\Phi$, the family of all functions $\psi, \phi : [0,\infty) \to \mathbb{R}$ that satisfing the conditions of above theorem.*

*Also, we denoted by $\mathcal{H}$, the set of all functions $H : [0,\infty)^3 \to [0,\infty)$ that satisfing the following conditions.*

$(H_1)$ $\max\{\alpha, \beta\} \le H(\alpha, \beta, \eta)$, *for all $\alpha, \beta, \eta \in [0,\infty)$.*
$(H_2)$ $H(0,0,0) = 0$.
$(H_3)$ *$H$ is continuous.*

As simple examples, the following mappings can be mentioned.

$$H(\alpha,\beta,\eta) = \alpha + \beta + \eta \quad H(\alpha,\beta,\eta) = \max\{\alpha,\beta\} + \eta,$$

for all $\alpha, \beta, \eta \in [0,\infty)$.

Let $\varphi : X \to [0,\infty)$ and $T : X \to X$. A point $x \in X$ is called $\varphi$- fixed point $T$, if $x \in F_T$ and $\varphi(x) = 0$.

Jleli et al. [?] introduced the concepts of $(H,\varphi)$-contraction and $(H,\varphi)$-weak contraction using the mapping $H$, and proved that these contractions have at least one $\varphi$-fixed point.

Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. In the following, this paper uses the following notations and concepts.

$$d(A,B) = \inf\{d(x,y); x \in A, y \in B\},$$

$$A_0 = \{x \in A : d(x,y) = d(A,B), y \in B\},$$

$$B_0 = \{y \in B : d(x,y) = d(A,B), x \in A\},$$

It should be noted that the above sets are not always nonempty. For example, Let $A = \{(x, x\sin\frac{1}{x}); x \in (0,1]\}$ and $B = \{(0,0)\}$. Then $d(A,B) = 0$ and $A_0 = \emptyset$.

The set of all best proximity points of a non-self mapping $T : A \to B$ is denoted by

$$B_{\text{est}}(T) = \{x \in A : d(x,Tx) = d(A,B)\}.$$

Also, the set of all zeros of the function $\varphi : A \to [0, \infty)$ is shown by $Z_\varphi$. In other words

$$Z_\varphi = \{x \in A; \varphi(x) = 0\}.$$

In 2017, Isik and colleagues, inspired by the above topics, presented the following definitions.

**Definition 1.3.** [9] *An element $x^* \in A$ is called a $\varphi$-best proximity point of the non-self mapping $T : A \to B$ if $x^* \in B_{est}(T) \cap Z_\varphi$.*

**Definition 1.4.** [9] *Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$ and $\varphi : A \to [0, \infty)$ be a given function. Also $H \in \mathcal{H}$. Then:*

(1) *A mapping $T : A \to B$ is called an $(H, \varphi)$-proximal contraction if there exists $k \in (0, 1)$ such that for every $u, v, x, y \in A$, we have:*

$$\begin{cases} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{cases} \Rightarrow H(d(u, v), \varphi(u), \varphi(v)) \leq k(H(d(x, y), \varphi(x), \varphi(y))).$$

(2) *A mapping $T : A \to B$ is called a weak $(H, \varphi)$-proximal contraction if there exist $k \in (0, 1)$ and $L \geq 0$ such that for every $u, v, x, y \in A$, we have:*

$$\begin{cases} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{cases} \Rightarrow$$

$$H(d(u, v), \varphi(u), \varphi(v)) \leq k(H(d(x, y), \varphi(x), \varphi(y))) + L[H(d(y, u), \varphi(y), \varphi(u)) - H(d(0), \varphi(y), \varphi(u))].$$

Isik and colleagues were able to establish conditions under which the mappings satisfying the above definition have a $\varphi$-best proximity point.

## 2. main context

In this section, we first define the concept of $(H, \psi, \varphi, \phi)$-proximal contraction, and then, by imposing certain conditions, we prove the existence and uniqueness of the $\varphi$-best proximity point.

**Definition 2.1.** *Let $A$ and $B$ be two nonempty subsets of a metric space $(X, d)$. We consider functions $\varphi : A \to [0, \infty)$ and $H \in \mathcal{H}$. Mapping $T : A \to B$ is called an $(H, \psi, \varphi, \phi)$-proximal contraction if there exist functions $\psi \in \Psi$ and $\phi \in \Phi$ such that for every $u, v, x, y \in A$, we have:*

$$\begin{cases} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{cases} \Rightarrow \psi(H(d(u, v), \varphi(u), \varphi(v))) \leq \phi(H(d(x, y), \varphi(x), \varphi(y))), \tag{2.1}$$

*where $H(d(x, y), \varphi(x), \varphi(y)) > 0$.*

**Theorem 2.2.** *Let $A$ and $B$ be two nonempty subset of metric space $(X, d)$ and non-self mapping $T : A \to B$ satisfies in the following conditions:*


http://dx.doi.org/10.22108/msci.2025.143009.1706

1) $A_0$ *be nonempty and closed.*
2) $T(A_0) \subseteq B_0$.
3) $\varphi : A \to [0, \infty)$ *be lower semi-continuous.*
4) $T$ *be a* $(H, \psi, \varphi, \phi)$*-proximal contraction.*

*then* $T$ *has a unique* $\varphi$*-best proximity point* $x^*$*. Moreover for every* $x \in X$*, we have* $T^n x \to x^*$*.*

If in Theorem 2.2 we have, $H(\alpha, \beta, \eta) = \alpha + \beta + \eta$, and $\varphi(x) = 0$, then we get the following result:

**Corollary 2.3.** *Let* $A$ *and* $B$ *be nonempty subsets of a complete metric space* $(X, d)$ *and the mapping* $T : A \to B$ *satisfies the following relation:*

$$\begin{cases} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{cases} \Rightarrow \psi(d(Tx, Ty)) \leq \phi(d(x, y)),$$

*where* $d(x, y) > 0$*,* $\psi \in \Psi$*,* $\phi \in \Phi$ *for all* $u, v, x, y \in A$*. If conditions (1), (2), and (3) of Theorem 2.2 hold, then* $T$ *has a unique* $\varphi$*-best proximity point.*

If for $t \in \mathbb{R}^+$ and $x \in X$ we have $\varphi(x) = 0$, $\psi(t) = t$ and $\phi(t) = kt$ where $k \in (0, 1)$. In this case, the following result follows from Theorem 2.2.

**Corollary 2.4.** *Let* $A$ *and* $B$ *be nonempty subsets of a complete metric space* $(X, d)$*. We consider the mapping* $T : A \to B$ *such that the following relation holds:*

$$\begin{cases} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{cases} \Rightarrow d(Tx, Ty) \leq kd(x, y),$$

*where* $d(x, y) > 0$*,* $k \in (0, 1)$*, for every* $u, v, x, y \in A$*, and conditions (1),(2) and (3) of Theorem 2.2 hold, then* $T$ *has a unique best proximity point.*

**Corollary 2.5.** *Let* $A$ *and* $B$ *be nonempty subsets of the complete metric space* $(X, d)$*, and let* $T : A \to B$ *be a mapping. Assume there exists* $\tau > 0$ *such that the following relation holds:*

$$\psi(H(d(Tx, Ty), \varphi(Tx), \varphi(Ty))) \leq \psi(H(d(x, y), \varphi(x), \varphi(y))) - \tau,$$

*where* $H(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) > 0$*, for every* $x, y \in A$*. If* $\psi \in \Psi$ *and conditions (1), (2), and (3) of Theorem 2.2 hold, then* $T$ *has a unique* $\varphi$*-best proximity point.*

If we set $A = B$ in Theorem 2.2, then we get the following results:

**Corollary 2.6.** *Let* $(X, d)$ *be a complete metric space and the mapping* $T : X \to X$ *satisfies the following condition:*

$$\psi(H(d(Tx, Ty), \varphi(Tx), \varphi(Ty))) \leq \phi(H(d(x, y), \varphi(x), \varphi(y))),$$

*where* $\psi \in \Psi$ , $\phi \in \Phi$ *and* $H(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) > 0$ *for all* $x, y \in X$. *If* $\varphi : X \to [0,\infty)$, *is lower semi-continuous then* $T$ *has a unique* $\varphi$*-fixed point* $x^*$. *Moreover, for every* $x \in X$, *we have* $lim_{n\to\infty} T^n x = x^*$.

**Corollary 2.7.** *Let* $(X, d)$ *be a complete metric space and let* $T : X \to X$ *be a mapping. Assume that there exists* $k \in [0, 1)$ *such that the following condition holds:*

$$H(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) > 0$$
$$\implies \psi(H(d(Tx,Ty),\varphi(Tx),\varphi(Ty))) \le \psi(H(d(TX,Ty),\varphi(Tx),\varphi(Ty)))^k,$$

*where* $\varphi : X \to [0,\infty)$ *is lower semi-continuous and* $\psi : \mathbb{R}^+ \to \mathbb{R}$ *is non-decreasing. Then,* $T$ *has a unique* $\varphi$*-fixed point, denoted by* $x^*$. *Moreover, for every* $x \in X$, *we have* $lim_{n\to\infty} T^n(x) = x^*$.

**Example 2.8.** *Consider the metric space* $X = [0, 1] \times [0, 1]$ *equipped with the Euclidean metric. Assume that*

$$A = \{(1 - \frac{1}{n}, 1); n \in \mathbb{N}, \}$$

*and*

$$B = \{(0, 0), (\frac{1}{2}, 0), (1, 0)\}.$$

*Then* $A_0 = \{(0, 1), (\frac{1}{2}, 1)\}$ *and* $B_0 = \{(0, 0), (\frac{1}{2}, 0)\}$ *and* $d(A, B) = 1$.
*We define the mapping* $T : A \to B$ *as follows:*

$$T(x, 1) = \begin{cases} (\frac{1}{2}, 0) & x \in [\frac{2}{3}, 1), \\ (0, 0) & x = 0 \ or \ \frac{1}{2}, \end{cases}$$

*We define the functions* $\psi, \phi : [0,\infty) \to \mathbb{R}$, $\varphi : A \to [0,\infty)$, $H : [0,\infty) \to [0,\infty)$ *as follows:*

$$\psi(x) = \frac{9}{10}x, \quad \phi(x) = x, \quad \varphi((x, 1)) = x, \quad H(\alpha, \beta, \eta) = \alpha + \beta + \eta.$$

*We show that the conditions of Theorem 2.2 are satisfied. We can write:*

$$\begin{cases} d((u, 1), T(x, 1)) = d(A, B) = 1, \\ d((v, 1), T(y, 1)) = 1. \end{cases} \tag{2.2}$$

*We examine the following situations:*

(1) *if* $x = y = \frac{1}{2}$, *then*

$$\begin{cases} d((u, 1), (0, 0)) = 1 \Rightarrow u = 0, \\ d((v, 1), (0, 0)) = 1 \Rightarrow v = 0. \end{cases}$$

*So, we have clearly:*

$$\psi(H(d((0, 1), (0, 1))), \varphi((0, 1)), \varphi((0, 1)))) \le \phi(H(d((x, 1), (y, 1)), \varphi((x, 1)), \varphi((y, 1)))).$$


http://dx.doi.org/10.22108/msci.2025.143009.1706

(2) *if* $x, y \in [\frac{2}{3}, 1)$, *then by* (2.2) *we have:*

$$\begin{cases} d((u,1),(\frac{1}{2},0)) = 1 \Rightarrow u = \frac{1}{2}, \\ d((v,1),(\frac{1}{2},0)) = 1 \Rightarrow v = \frac{1}{2}. \end{cases}$$

*Hence*

$$\psi(H(d((\frac{1}{2},1),(\frac{1}{2},1)),\varphi((\frac{1}{2},1)),\varphi((\frac{1}{2},1)))) \leq \phi(H(d((x,1),(y,1)),\varphi((x,1)),\varphi((y,1)))),$$

*so*

$$\frac{1}{2} + \frac{1}{2} \leq \frac{9}{10}(\frac{2}{3} + \frac{2}{3}) \leq d((x,1),(y,1)) + x + y.$$

(3) *if* $x = \frac{1}{2}$ *and* $y \in [\frac{2}{3}, 1)$, *then by* (2.2) *we have:*

$$\begin{cases} d((u,1),(0,0)) = 1 \Rightarrow u = 0, \\ d((v,1),(\frac{1}{2},0)) = 1 \Rightarrow v = \frac{1}{2}. \end{cases}$$

*Since* $\frac{1}{2} + \frac{1}{2} \leq \frac{9}{10}(\frac{2}{3} + \frac{2}{3}) \leq \frac{9}{10}(2y)$ *so:*

$$\psi(H(d((0,1),(\frac{1}{2},1)),\varphi((0,1)),\varphi(\frac{1}{2},1)))) \leq \phi(H(d((\frac{1}{2},1),(y,1)),\varphi((\frac{1}{2},1),\varphi((y,1)))).$$

*For other cases, relation* (2.1) *also holds. Therefore, all conditions of Theorem* 2.2 *hold and consequently the mapping* $T$ *has a* $\varphi$*-best proximity points* $(0,1)$.

**Example 2.9.** *Let* $X = \{0, \frac{1}{2}, \frac{1}{3}, \ldots, 1\}$, $A = \{0, \frac{1}{2}, \frac{1}{4}, \ldots, 1\}$ *and* $B = \{0, \frac{1}{3}, \frac{1}{5}, \ldots, 1\}$.

*We define functions* $d : X \times X \to \mathbb{R}$ *and* $T : A \to B$ *as follow:*

$$d(x,y) = \begin{cases} 0 & x = y \\ \max\{x,y\} & x \neq y, \end{cases} \qquad Tx = \frac{x}{x+1},$$

*Let the functions* $\psi, \phi : [0,\infty) \to \mathbb{R}$, $\varphi : A \to [0,\infty)$ *and* $H : [0,\infty) \to [0,\infty)$ *is defined as follows:*

$$\psi(t) = t, \quad \phi(t) = \frac{1}{2}t, \quad \varphi((x,1)) = x, \quad H(\alpha,\beta,\eta) = \alpha + \beta + \eta.$$

*Then* $d(A,B) = 0$ *and* $A_0 = \{0\} = B_0$. *So*

$$\begin{cases} d(u,Tx) = d(A,B) = 0 \Rightarrow u = Tx, \\ d(v,Ty) = d(A,B) = 0 \Rightarrow v = Ty. \end{cases}$$

*Therefore* $u = v = Tx = Ty = 0$. *Hence* $x = y = 0$. *Clearly, all the conditions of Theorem* 2.2 *hold and consequently the function* $T$ *has a* $\varphi$*-best proximity points* $x = 0$.

Here we assume the functions $\psi, \phi : [0,\infty) \to \mathbb{R}$ are non-decreasing and lower semi-continuous and also

$$\psi(t) = \phi(t) = 0 \Leftrightarrow t = 0.$$

**Definition 2.10.** *Let $A$ and $B$ be non-negative subsets of the metric space $(X,d)$. Consider the function $\varphi : A \to [0,\infty)$ and $H \in \mathcal{H}$. The mapping $T : A \to B$ is called a weakly proximal $(H,\psi,\varphi,\phi)$-weak proximal contraction if there exist functions $\psi$ and $\phi$ satisfying the above conditions, and we have:*

$$\begin{cases} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{cases} \Rightarrow$$

$$\psi(H(d(u,v),\varphi(u),\varphi(v))) \leq \psi(H(d(x,y),\varphi(x),\varphi(y))) - \phi(H(d(x,y),\varphi(x),\varphi(y)),$$

*when $H(d(x,y),\varphi(x),\varphi(y)) > 0$ for every $u,v,x,y \in A$.*

**Theorem 2.11.** *Let $A$ and $B$ be nonempty subsets of a complete metric space $(X,d)$ and a non-self mapping $T : A \to B$ satisfies the following conditions:*

1) *$A_0$ be a nonempty and closed.*
2) *$T(A_0) \subseteq B_0$.*
3) *$\varphi : A \to [0,\infty)$ be a lower semi-continuous.*
4) *$T$ be a $(H,\psi,\varphi,\phi)$-weak proximal contraction.*

*Then $T$ has a unique $\varphi$-best proximity point $x^*$. Moreover, for all $x \in X$ we have $T^n x \to x^*$.*

Let for $t \in \mathbb{R}^+$ we have $\psi(t) = t$ and $\phi(t) = (1-k)t$, when $k \in (0,1)$. Then the following results is obtained from 2.11.

**Corollary 2.12.** *Let $A$ and $B$ be nonempty subsets of complete metric space $(X,d)$. We consider the functions $\varphi : A \to [0,\infty)$ and $H \in \mathcal{H}$. If the mapping $T : A \to B$ satisfies in the following condition:*

$$\begin{cases} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{cases} \Rightarrow H(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \leq k(H(d(x,y),\varphi(x),\varphi(y))),$$

*when $H(d(x,y),\varphi(x),\varphi(y)) > 0$, $k \in (0,1)$ for all $u,v,x,y \in A$ and conditions (1),(2) and (3) of Theorem 2.11 hold, then $T$ has a unique $\varphi$-best proximity point.*

**Corollary 2.13.** *Let $(X,d)$ be a complete metric space and $T : X \to X$ satisfies in the follow condition:*

$$\psi(H(d(Tx,Ty),\varphi(Tx),\varphi(Ty))) \leq \psi(H(d(x,y),\varphi(x),\varphi(y))) - \phi(H(d(x,y),\varphi(x),\varphi(y))),$$

*where $H(d(x,y),\varphi(x),\varphi(y)) > 0$ for all $x,y \in X$. Then $T$ has a $\varphi$- unique fixed point $x^*$. Moreover, for all $x \in X$ we have $\lim_{n\to\infty} T^n x = x^*$.*

 http://dx.doi.org/10.22108/msci.2025.143009.1706

**Example 2.14.** *We consider metric space* $X = \{1, 2, 3, 4, 5\}$ *with Euclidean metric and subsets* $A = \{1, 3\}$ *and* $B = \{1, 2, 4\}$*. So* $A_0 = \{1\}$*,* $B_0 = \{1\}$ *and* $d(A, B) = 0$*. Now let the mapping* $T$ *define as* $Tt = \frac{t+1}{2}$*. We consider functions* $\psi, \phi : [0, \infty) \to \mathbb{R}$*,* $\varphi : A \to [0, \infty)$*,* $H : [0, \infty) \to [0, \infty)$ *as follows:*

$$\psi(t) = t, \quad \phi(t) = \frac{1}{2}t, \quad \varphi(t) = \ln t, \quad H(\alpha, \beta, \eta) = \alpha + \beta + \eta.$$

*We show that conditions of Theorem 2.11 are hold. It can be seen that:*

$$\begin{cases} d(u, Tx) = d(A, B) = 0, \\ d(v, Ty) = d(A, B) = 0. \end{cases}$$

*So* $u = v = x = y = 1$*. Therefore:*

$$\psi(H(d(u, v), \varphi(u), \varphi(v))) \leq \psi(H(d(x, y), \varphi(x), \varphi(y))) - \phi(H(d(x, y), \varphi(x), \varphi(y))).$$

*As a result, the mapping* $T$ *is a weakly contractive non-self-map and by Theorem 2.11 the mapping* $T$ *has a* $\varphi$*-best proximity point.*

## 3. Conclusions

In this paper, we introduce a new concept of proximal mappings called $(H, \psi, \varphi, \phi)$-proximal contraction and $(H, \psi, \varphi, \phi)$-weak proximal contraction. Then, we prove theorems regarding the existence and uniqueness of a $\phi$-best proximity point for these mappings and demonstrate the effectiveness of our results by providing examples.

## Acknowledgments

This work has been financially supported by the research deputy of Shahrekord University. The grant number was 3GRN1M1024.

## References

[1] A. Amini-Harandi, Common best proximity points theorems in metric spaces, *Optim. lett.*, **8** no. 2 (2014) 581–589.

[2] M. A. Al-Thagafi and N. Shahzad, Convergence and existence results for best proximity points, *Nonlinear Anal.*, **70** no. 10 (2009) 3665–3671.

[3] M. Cosentino and P. Vetro, Fixed point results for $F$-contractive mappings of Hardy-Rogers-type, *Filomat*, **28** no. 4 (2014) 715–722.

[4] A. A Eldred and P. Veeramani, Existence and convergence of best proximity points, *J. Math. Anal. Appl.*, **323** no. 2 (2006) 1001–1006.

[5] M. Gabeleh, Semi-normal structure and best proximity pair results in convex metric spaces, *Banach J. Math. Anal.*, **8** no. 2 (2014) 214–228.

[6] M. Gabeleh, Best proximity points and fixed point results for certain maps in Banach spaces, *Numer. Funct. Anal. Optim.*, **36** no. 8 (2015) 1013–1028.

[7] M. Gabeleh and J. Markin, A note on the paper Best proximity point results for $p$-proximal contractions, *Acta Math. Hungar.*, **164** no. 1 (2021) 326–329.

[8] M. Gabeleh and C. Vetro, A note on best proximity point theory using proximal contractions, *J. Fixed Point Theory Appl.*, **20** no. 4 (2018) 11 pp.

[9] H. IŞik, M. S. Sezen and C. Vetro, $\phi$-best proximity point theorems and applications to variational inequality problems, *J. Fixed Point Theory Appl.*, **19** no.4 (2017) 3177–3189.

[10] M. Jleli, B. Samet and C. Vetro, Fixed point theory in partial metric spaces via $\varphi$-fixed point's concept in metric spaces, *J. Inequal. Appl.*, **2014** (2014) 9 pp.

[11] X. D. Liu, S. S. Chang, Y. Xiao and L. C. Zhao, Some fixed point theorems concerning $(\psi, \phi)$-type contraction in complete metric spaces, *J. Nonlinear Sci. Appl.*, **9** no. 6 (2016) 4127–4136.

[12] P. D. Proinov, Fixed point theorem for generalized contractive mappings in metric spaces, *J. Fixed Point Theory Appl.*, **22** no. 1 (2020) 27 pp.

[13] V. Sankar Raj, A best proximity point theorem for weakly contractive non-self-mappings, *Nonlinear Anal.*, **74** no. 14 (2011) 4804–4808.

[14] M. Shams, S. Zamani, S. Jafari and M. De La Sen, Existence of $\varphi$-fixed point for generalized contractive mappings, *AIMS Math.*, **6** no. 7 (2021) 7017-7033.

[15] W. Sintunavarat and P. Kumam, Coupled best proximity point theorem in metric spaces, *Fixed Point Theory Appl.*, **2012** (2012) 16 pp.

[16] T. Suzuki, M. Kikkawa and C. Vetro, The existence of best proximity points in metric spaces with the property UC, *Nonlinear Anal.*, **71** no. 7-8 (2009) 2918-2926.

[17] D. Wardowski, Fixed points of new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012** (2012) 6 pp.

[18] D. Wardowski and N. V. Dung, Fixed points of $F$-weak contractions on complete metric spaces, *Demonstr. Math.*, **47** no. 1 (2014) 146–155.

[19] J. Zhang and Y. Su, Best proximity point theorems for weakly contractive mapping and weakly Kannan mapping in partial metric spaces, *Fixed Point Theory APPL.*, **2014** (2014) 8 pp.

**Maryam Shams**

Department of Mathematical Sciences, Shahrekord University, Shahrekord, Iran

Email:maryam.shams@sku.ac.ir

**Farah Rashidi**

Department of Mathematical Sciences, Shahrekord University, Shahrekord, Iran

Email:rashidi.farah66@yahoo.com

 http://dx.doi.org/10.22108/msci.2025.143009.1706