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EQUIVARIANT T-HARMONIC MAPS BETWEEN COMPACT RIEMANNIAN MANIFOLDS OF COHOMOGENEITY ONE

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Abstract. In this paper, Urakawa's work on equivariant harmonic maps between compact Riemannian manifolds with codimension one is extended to equivariant T-harmonic maps between them. The Euler-Lagrange equation for the harmonicity of these maps has been transformed into their T-harmonicity. It is assumed that both the domain and target Riemannian manifolds have codimension one, meaning they both have isometry group actions with orbits of codimension one. Then, the ordinary differential equations for equivariant T-harmonic maps between them have been derived. As an application, T-harmonic maps from 2-dimensional flat tori into spheres have been constructed. All equivariant T-harmonic maps from a flat torus into a Riemannian manifold that admits a codimension one action of a compact Lie group have been identified by solving a system of ordinary differential equations.

1. Introduction

The aim of this paper is to develop works of Urakawa [28] on equivariant harmonic maps between compact Riemannian manifolds of cohomogeneity one, to equivariant T-harmonic maps between them, which is introduced in [1] (see also [2]), and also his reduction of the Euler-Lagrange equation on

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harmonicity of these maps to T-harmonicity of them. As applications, we construct T-harmonic maps from 2-flat tori into spheres.

To construct T-harmonic maps, We assume that both domain and target Riemannian manifolds are of cohomogeneity 1; that is, both admit the isometry group actions having orbits of codimension 1. Then we derive the ordinary differential equations of the equivariant T-harmonic maps between them.

We recall the prerequisites from [1, 2, 27, 28]. Let (M, g) be a compact Riemannian manifold and K a compact Lie group acting effectively and isometrically on M with cohomogeneity one; that is, there is an orbit of codimension one, so $\dim(Kx) = \dim M - 1$ for some $x \in M$. The orbit space M/K is either a closed interval $[0, l]$ or a circle. Focusing on the interval case, we describe the structure of K and (M, g), following [28]. Let $c(t)$, $0 < t < l$, be a geodesic in M representing M/K. The isotropy subgroup at $c(t)$ is J_t, which is constant J for $t \in (0, l)$. The Lie algebra $\mathtt{k}$ of K admits an orthogonal decomposition relative to an $\mathrm{Ad}(K)$-invariant inner product $\langle , \rangle$:

$$\mathtt{k} = \mathfrak{j} \oplus \mathfrak{m},$$

where $\mathfrak{j}$ is the Lie algebra of J, and $\mathfrak{m}$ is $\mathrm{Ad}(J)$-invariant.

The map $K/J \times [0, l] \to M$ defined by $(kJ, t) \mapsto kc(t)$ is surjective; restricted to $K/J \times (0, l)$ it is smooth and its image $\mathring{M}$ is an open dense subset of M. On $\mathring{M}$, the metric can be expressed as

$$g = dt^2 + g_t,$$

where g_t is the K-invariant metric on the orbit $Kc(t)$. For $X, Y \in \mathfrak{m}$,

$$g_t(X_{c(t)}, Y_{c(t)}) = \alpha_t(X, Y),$$

with X extended to a vector field on M by $X_p = \frac{d}{ds}\big|_{s=0} \exp(sX) \cdot p$. The inner product α_t on $\mathfrak{m}$ is diagonal with respect to an orthonormal basis $\{X_i\}_{i=1}^{m-1}$:

$$\alpha_t(X_i, X_j) = f_i(t)^2 \delta_{ij}, \quad 1 \leq i, j \leq m - 1,$$

where $m = \dim M$. Finally, an orthonormal frame $\{e_i\}_{i=1}^{m}$ near $c(t)$ is constructed on

$$W := \{kc(s) \mid k \in U \subset \exp(\mathfrak{m}), |s - t| < \varepsilon\}$$

by

$$\begin{cases} (e_i)_{kc(s)} := f_i(s)^{-1} \tau_{k*} X_{ic(s)}, & 1 \leq i \leq m - 1, \\ (e_m)_{kc(s)} := \tau_{k*} \dot{c}(s), \end{cases} \tag{1.1}$$

where τ_k denotes the action of $k \in K$ on M, $\dot{c}(s)$ is the tangent vector to c at s, and U is a small neighborhood of e in $\exp(\mathfrak{m})$.

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Definition 1.1. [1] *Let $T : \mathcal{X}(M) \to \mathcal{X}(M)$ be a smooth tensor on M, $\phi : (M, g) \to (N, h)$ be a smooth map. The differential operator $\overset{\mathbf{T}}{\square}$ is defined as follow:*

$$\overset{\mathbf{T}}{\square}(\phi) = \sum_{i=1}^{m} (\tilde{\nabla}_{Te_i}\phi_*(e_i) - \phi_*(\nabla_{Te_i}e_i)).$$

Theorem 1.2. [1] *The map ϕ is T-harmonic map if and only if*

$$\overset{\mathbf{T}}{\square}(\phi) + \frac{1}{2}\phi_*\left(\operatorname{div}(T + T^t)\right) = 0\,. \tag{1.2}$$

The L.H.S of equation (1.2), is called Amin-tension field and denoted by

$$A_T(\phi) = \overset{\mathbf{T}}{\square}(\phi) + \frac{1}{2}\phi_*\left(\operatorname{div}(T + T^t)\right),$$

which is a generalization of the notion introduced in [2].

2. Main Results

Consider compact Riemannian manifolds (M, g) and (N, h), each admitting effective, isometric cohomogeneity-one actions by compact Lie groups K and G, respectively. Their orbit spaces can be identified as intervals $M/K = [0, l]$ and $N/G = [0, \bar{l}]$, with corresponding geodesics $c(t)$ on M for $t \in [0, l]$ and $\bar{c}(r)$ on N for $r \in [0, \bar{l}]$. Denote the isotropy subgroups at these points by $J_t \subset K$ and $H_r \subset G$. For $0 < t < l$ and $0 < r < \bar{l}$, these stabilize to fixed groups J and H, respectively [28].

Let $A : K \to G$ be a Lie group homomorphism. A map $\phi : M \to N$ is called A-equivariant if it satisfies $\phi(kx) = A(k)\phi(x)$ for all $k \in K, x \in M$. Such an ϕ induces a function $r : [0, l] \to [0, \bar{l}]$ and a map $\Psi : [0, l] \to G$ with

$$\phi(c(t)) = \Psi(t)\bar{c}(r(t)), \quad t \in [0, l].$$

The A-equivariance condition requires

$$\Psi(t)^{-1}A(J_t)\Psi(t) \subset H_{r(t)}, \quad t \in [0, l]. \tag{2.1}$$

Conversely, any pair (r, Ψ) satisfying the condition (2.1) defines an A-equivariant map by

$$\phi(kc(t)) = A(k)\Psi(t)\bar{c}(r(t)), \quad k \in K,\ t \in [0, l].$$

Moreover, all A-equivariant maps from M to N arise in this manner.

Since G act isometrically on (M, g) , we get $\{\tau_{k*}e_i\}_{i=1}^m$ a local orthonormal frame field on M and

$$\overset{\mathbf{T}}{\square}(\phi)(kx) = \sum_i (\tilde{\nabla}_{T\tau_{k*}e_i}\phi_*(\tau_{k*}e_i) - \phi_*(\nabla_{T\tau_{k*}e_i}\tau_{k*}e_i)).$$

Throughout the paper, assume that $T \circ \tau_{k*} = \tau_{k*} \circ T$. Therefore

$$\overset{\mathbf{T}}{\square}(\phi)(kx) = \sum_i (\tilde{\nabla}_{\tau_{k*}Te_i}\phi_*(\tau_{k*}e_i) - \phi_*(\nabla_{\tau_{k*}Te_i}\tau_{k*}e_i)).$$

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Since $\phi \circ \tau_k = \tau_{A(k)} \circ \phi,\ k \in K$, we have $\phi_* \circ \tau_{k*} = \tau_{A(k)*} \circ \phi_*$, and by isometry of $\tau_{A(k)}$,

$$\begin{aligned}\overset{\mathbf{T}}{\square}(\phi)(kx) &= \sum_i \left(\tilde{\nabla}_{\tau_{k*}Te_i}\phi_*\left(\tau_{k*}(e_i)\right) - \phi_*\tau_{k*}\nabla_{Te_i}e_i\right)\\ &= \sum_i \left(\tilde{\nabla}_{\tau_{k*}Te_i}\tau_{A(k)*}\left(\phi_*(e_i)\right) - \tau_{A(k)*}\circ\phi_*\nabla_{Te_i}e_i\right)\\ &= \tau_{A(k)*}\overset{\mathbf{T}}{\square}(\phi)(x),\end{aligned}$$

where $k \in K,\ x \in \mathring{M}$. Then we only have to calculate $\overset{\mathbf{T}}{\square}(\phi)$ at $c(t),\ 0 < t < l$.

To proceed, recall that the metrics g and h on M and N are described as follows: Let $\mathfrak{m} \subset \mathfrak{k}$ and $\mathfrak{n} \subset \mathfrak{g}$ be the subspaces invariant under $\mathrm{Ad}(J)$ and $\mathrm{Ad}(H)$, respectively, and orthogonal to $\mathfrak{j}$ and $\mathfrak{h}$ with respect to the inner products $\langle,\rangle$ on $\mathfrak{k}$ and $\mathfrak{g}$. Then

$$g = dt^2 + g_t,\ \ h = dr^2 + h_r,$$

and the inner products α_t, and β_r, on $\mathfrak{m}$ and $\mathfrak{n}$ are induced from g_t, and h_r. Choose orthonormal bases $\{X_j\}_{j=1}^{m-1}$ and $\{Y_a\}_{a=1}^{n-1}$ of $(\mathfrak{m}, \langle,\rangle)$ and $(\mathfrak{n}, \langle,\rangle)$ in such a way that

$$\alpha_t(X_i, X_j) = f_i(t)^2\delta_{ij} \quad \text{and} \quad \beta_r(Y_a, Y_b) = h_a(r)^2\delta_{ab}.$$

As in formula (1.1), define orthonormal frame fields $\{e_j\}_{j=1}^{m-1}$ and $\{\bar{e}_a\}_{a=1}^{n-1}$ on neighborhoods W and $\bar{W}$ of $c(t)$ and $\bar{c}(r)$, respectively. Then we obtain the following proposition.

Proposition 2.1. *Assume that the function $r(t) : [0, l] \to [0, \bar{l}]$ satisfies $r(0) = 0,\ r(l) = \bar{l}$, and $0 < r(t) < \bar{l}$ for $0 < t < l$. Then $\overset{\mathbf{T}}{\square}(\phi)$ which $\phi : (M, g) \to (N, h)$ is an A-equivariant map and smooth on $\mathring{M}$, can be described as*

$$\begin{aligned}\overset{\mathbf{T}}{\square}(\phi)(c(t)) &= \sum_{a=1}^{n-1}\sum_{i,j=1}^{m-1} T_{ij}(c(t))\Big(f_i(t)^{-1}f_j(t)^{-1}h_a(r)^{-1}\beta_r(Y_a,[V_i,U_j])\Big)\bar{e}_a\\ &+\sum_{a=1}^{n-1}\sum_{j=1}^{m-1}(T_{jm}+T_{mj})\Big(\ \dot{r}(t)\ h_a(r)^{-2}f_j(t)^{-1}\big(\frac{d}{dr}h_a(r)\big) - f_j(t)^{-2}\big(\frac{d}{dt}f_j(t)\big)h_a(r)^{-1}\Big)\beta_r(Y_a,U_j)\bar{e}_a\\ &+\sum_{a=1}^{n-1}\sum_{j=1}^{m-1}T_{mj}(c(t))h_a(r)^{-1}f_j(t)^{-1}\beta_r(Y_a,\frac{d}{dt}U_j)\bar{e}_a\\ &+\Bigg(T_{mm}(c(t))\ddot{r}(t) + \sum_{i=1}^{m-1}T_{ii}\ f_i(t)^{-1}\frac{df_i}{dt}\dot{r}(t)\\ &-\sum_{a=1}^{n-1}\sum_{i,j=1}^{m-1}T_{ij}\ h_a(r)^{-3}\big(\frac{d}{dr}h_a(r)\big)f_i(t)^{-1}f_j(t)^{-1}\beta_r(Y_a,U_i)\beta_r(Y_a,U_j)\Bigg)\bar{e}_n\end{aligned}$$

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$$+\sum_{i,j=1}^{m-1} T_{ij}\ f_i(t)^{-1} f_j(t)^{-1} \tau_{\Psi *}\left\{\frac{1}{2}[U_i,U_j]+V_r(U_i,U_j)-\left(\mathrm{Ad}(\Psi^{-1})A\Big(\frac{1}{2}[X_i,X_j]_{\mathfrak{m}}+U_t(X_i,X_j)\Big)\right)_{\mathfrak{n}}\right\}_{\bar{c}(r)},$$

for $r=r(t)$, $\Psi=\Psi(t)$, $0<t<l$. Here we put $T_{ij}=\langle T(e_j),e_i\rangle_g$ where $\{e_i\}_{i=1}^m$ is an orthonormal frame field given by equations (1.1), $A(X_j)=\mathrm{Ad}(\Psi)(U_j+V_j)$, $U_j\in\mathfrak{n}$, $V_j\in\mathfrak{h}$, and $X_{\mathfrak{n}}$ is the $\mathfrak{n}$-component of $X\in\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{n}$. Also $U_t(X,Y)\in\mathfrak{m}$ and $V_r(X',Y')\in\mathfrak{n}$ are defined by

$$2\alpha_t(U_t(X,Y),\ Z)=\alpha_t(X,\ [Z,Y]_{\mathfrak{m}})+\alpha_t([Z,X]_{\mathfrak{m}},\ Y),\qquad X,Y,Z\in\mathfrak{m};$$
$$2\beta_r(V_r(X',Y'),\ Z')=\beta_r(X',\ [Z',Y']_{\mathfrak{n}})+\beta_r([Z',X']_{\mathfrak{n}},\ Y'),\qquad X',Y',Z'\in\mathfrak{n}.$$

Now By Theorem 1.2 and Proposition 2.1 we get the following result.

Theorem 2.2. *(i) Assume that the function $r(t):[0,l]\to[0,\bar{l}]$ satisfies $r(0)=0$, $r(l)=\bar{l}$, and $0<r(t)<\bar{l}$ for $0<t<l$. Then the Amin-tension field $A_T(\phi)$ which $\phi:(M,g)\to(N,h)$ is an A-equivariant map and smooth on $\mathring{M}$, can be described as*

$$A_T(\phi)(c(t))=\sum_{a=1}^{n-1}\Bigg[\sum_{i,j=1}^{m-1}T_{ij}(c(t))\Big(f_i(t)^{-1}f_j(t)^{-1}h_a(r)^{-1}\beta_r(Y_a,[V_i,U_j])\Big)$$
$$+\sum_{j=1}^{m-1}(T_{jm}+T_{mj})\Big(\ \dot{r}(t)\ h_a(r)^{-2}f_j(t)^{-1}\big(\frac{d}{dr}h_a(r)\big)-f_j(t)^{-2}\big(\frac{d}{dt}f_j(t)\big)h_a(r)^{-1}\Big)\beta_r(Y_a,U_j)$$
$$+\sum_{j=1}^{m-1}T_{mj}(c(t))h_a(r)^{-1}f_j(t)^{-1}\beta_r(Y_a,\frac{d}{dt}U_j)$$
$$+\frac{1}{2}\sum_{j=1}^{m-1}\Bigg(\Big(\frac{d}{dt}\left(T_{jm}+T_{mj}\right)\Big)+\Big(\left(T_{mj}+T_{jm}\right)\ f_j(t)^{-1}\big(\frac{d}{dt}f_j(t)\big)\Big)+\sum_{i=1}^{m-1}f_i(t)^{-1}\frac{df_i}{dt}\left(T_{jm}+T_{mj}\right)$$
$$-\sum_{i,k=1}^{m-1}\left(T_{jk}+T_{kj}\right)f_i(t)^{-2}f_k(t)^{-1}\alpha_t\left(U_t(X_i,X_i),X_k\right)\Bigg)h_a(r)^{-1}f_j(t)^{-1}\beta_r(Y_a,U_j)\Bigg]\bar{e}_a$$
$$+\Bigg[T_{mm}(c(t))\ddot{r}(t)-\sum_{a=1}^{n-1}\sum_{i,j=1}^{m-1}T_{ij}\ h_a(r)^{-3}\big(\frac{d}{dr}h_a(r)\big)f_i(t)^{-1}f_j(t)^{-1}\beta_r(Y_a,U_i)\beta_r(Y_a,U_j)$$
$$+\Bigg(\frac{d}{dt}T_{mm}+\sum_{i=1}^{m-1}f_i(t)^{-1}\frac{df_i}{dt}T_{mm}$$
$$-\frac{1}{2}\sum_{i,j=1}^{m-1}\left(T_{mj}+T_{jm}\right)f_i(t)^{-2}f_j(t)^{-1}\alpha_t\left(U_t(X_i,X_i),X_j\right)\Bigg)\dot{r}(t)\Bigg]\bar{e}_n$$
$$+\sum_{i,j=1}^{m-1}T_{ij}\ f_i(t)^{-1}f_j(t)^{-1}\tau_{\Psi *}\left\{\frac{1}{2}[U_i,U_j]+V_r(U_i,U_j)-\frac{1}{2}\left(\mathrm{Ad}(\Psi^{-1})A\Big([X_i,X_j]_{\mathfrak{m}}\Big)\right)_{\mathfrak{n}}\right\}_{\bar{c}(r)}.$$

(ii) In particular, ϕ is T-harmonic map if and only if the following equations satisfied

$$\begin{aligned}
&\sum_{i,j=1}^{m-1} T_{ij}(c(t))\Big(f_i(t)^{-1}f_j(t)^{-1}h_a(r)^{-1}\beta_r(Y_a,[V_i,U_j])\Big)\\
&+\sum_{j=1}^{m-1}(T_{jm}+T_{mj})\Big(\ \dot r(t)\ h_a(r)^{-2}f_j(t)^{-1}\big(\frac{d}{dr}h_a(r)\big)-f_j(t)^{-2}\big(\frac{d}{dt}f_j(t)\big)h_a(r)^{-1}\Big)\beta_r(Y_a,U_j)\\
&+\sum_{j=1}^{m-1}T_{mj}(c(t))h_a(r)^{-1}f_j(t)^{-1}\beta_r(Y_a,\frac{d}{dt}U_j)\\
&+\frac{1}{2}\sum_{j=1}^{m-1}\Bigg(\Big(\frac{d}{dt}(T_{jm}+T_{mj})\Big)+\Big((T_{mj}+T_{jm})\ f_j(t)^{-1}\big(\frac{d}{dt}f_j(t)\big)\Big)+\sum_{i=1}^{m-1}f_i(t)^{-1}\frac{df_i}{dt}(T_{jm}+T_{mj})\\
&-\sum_{i,k=1}^{m-1}(T_{jk}+T_{kj})\,f_i(t)^{-2}f_k(t)^{-1}\alpha_t\left(U_t(X_i,X_i),X_k\right)\Bigg)h_a(r)^{-1}f_j(t)^{-1}\beta_r(Y_a,U_j)\\
&+\sum_{i,j=1}^{m-1}T_{ij}\ f_i(t)^{-1}f_j(t)^{-1}h_a(r)^{-1}\beta_r\left(\frac{1}{2}[U_i,U_j]+V_r(U_i,U_j)-\frac{1}{2}\Big(\mathrm{Ad}(\Psi^{-1})A\Big([X_i,X_j]_{\mathfrak{m}}\Big)\Big)_{\mathfrak{n}},Y_a\right)=0,
\end{aligned}$$

for $a=1,\dots,n-1$, and

$$\begin{aligned}
&T_{mm}(c(t))\ddot r(t)-\sum_{a=1}^{n-1}\sum_{i,j=1}^{m-1}T_{ij}\ h_a(r)^{-3}\big(\frac{d}{dr}h_a(r)\big)f_i(t)^{-1}f_j(t)^{-1}\beta_r(Y_a,U_i)\beta_r(Y_a,U_j)\\
&+\left(\frac{d}{dt}T_{mm}+\sum_{i=1}^{m-1}f_i(t)^{-1}\frac{df_i}{dt}T_{mm}-\frac{1}{2}\sum_{i,j=1}^{m-1}(T_{mj}+T_{jm})\,f_i(t)^{-2}f_j(t)^{-1}\alpha_t\left(U_t(X_i,X_i),X_j\right)\right)\dot r(t)=0.
\end{aligned}$$

3. Summary of Proofs/Conclusions

From this point on, we will consider an A-equivariant map ϕ satisfying the condition

$$\phi(c(t))=\bar c(r(t)),\ \text{that is, } \Psi\equiv 1. \tag{3.1}$$

and classify all A-equivariant T-harmonic maps satisfying the condition (3.1) from a flat torus into a Riemannian manifold (N,h). Let $(M,g)=(T^2,g)$ be a flat torus where

$$K=SO(2)=\left\{\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix};\theta\in\mathbb{R}\right\}$$

acts cohomogeneity 1. Then $T^2=R/2\pi\mathbb{Z}\times R/T\mathbb{Z}$ with some $T>0$. The constant $T>0$ will be determined as the period of the periodic solution of some ODE. In this case, the isotropy subgroup J_t of K consists only of the identity,

$$\mathfrak{m}=\mathbb{R}X_1 \text{ with } X_1=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \text{ and } f_1(t)\equiv 1\,.$$

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Let A be a homomorphism of $K = SO(2)$ with the compact Lie group G. Then any A-equivariant map ϕ with condition (3.1) of (T^2, g) into (N, h) is of the form

$$\phi(t,\, \theta) := \exp \theta A(X_1)\bar{c}(r(t)),$$

where $\bar{c}(r)$, $0 \leq r \leq \bar{l}$, is the representing geodesic of (N, h), and A satisfies always the condition (2.1).

Proposition 3.1. *Let A be a homomorphism $K = SO(2) \to G$ satisfying $A(X_1) = U_1 + V_1$, $U_1 \in \mathfrak{n}$, $V_1 \in \mathfrak{h}$, and $[U_1, V_1] = 0$. Then all the A-equivariant T-harmonic maps $\phi : (T^2, g) \to (N, h)$ with condition (3.1) which admit cohomogeneity 1 action of G are exhausted by all solutions $r(t)$ of ODE system*

$$(T_{12} + T_{21})\,(c(t))\;\dot{r}(t)\;\, h_a(r)^{-1}\big(\frac{d}{dr}h_a(r)\big)\beta_r(Y_a, U_1) + \frac{d}{dt}\left(\left(T_{12} + T_{21}\right)(c(t))\right)\beta_r(Y_a, U_1)$$
$$+\, T_{11}(c(t))\;\, \beta_r\left(V_r(U_1, U_1), Y_a\right) = 0,$$

for $a = 1, \ldots, n-1$, and

$$T_{22}(c(t))\ddot{r}(t) - \sum_{a=1}^{n-1} T_{11}(c(t))\;\, h_a(r)^{-3}\big(\frac{d}{dr}h_a(r)\big)\beta_r(Y_a, U_1)^2 + \dot{r}(t)\frac{d}{dt}(T_{22}(c(t))) = 0,$$

such that $\bar{c}(r(t))$ is periodic in t with period T, and the corresponding T-harmonic maps are given by

$$\phi(t,\, \theta) := \exp \theta U_1 \bar{c}(r(t)).$$

Proof. Since $\exp \theta A(X_1) = \exp \theta U_1 \exp \theta V_1$ due to $[U_1, V_1] = 0$ and $\exp \theta V_1 \bar{c}(r(t)) = \bar{c}(r(t))$, and by Theorem 2.2 we get the result. $\square$

Example 3.2. Consider the natural action of $G = SO(p+1) \times SO(n-p) \subset SO(n+1)$ on $\mathbb{S}^n$, (cf. [28]). In this case, the map ϕ in Proposition 3.1, is of the form

$$\phi(t,\, \theta) = \cos r(t) \exp \theta \begin{pmatrix} 0 & -X^t \\ X & 0 \end{pmatrix} \xi_1 + \sin r(t) \exp \theta \begin{pmatrix} 0 & -Y^t \\ Y & 0 \end{pmatrix} \xi_{p+2} \in \mathbb{S}^n,$$

where $\{\xi_j\}_{j=1}^{n+1}$ is the standard basis of $\mathbb{R}^{n+1}$, and $X \in \mathbb{R}^p$ and $Y \in \mathbb{R}^{n-p-1}$ are arbitrary vectors such that both matrices

$$\exp \theta \begin{pmatrix} 0 & -X^t \\ X & 0 \end{pmatrix} \quad \text{and} \quad \exp \theta \begin{pmatrix} 0 & -Y^t \\ Y & 0 \end{pmatrix},$$

are periodic in θ with period 2π. The function $r(t)$ is a solution of ODE system

$$\begin{cases} \Big(-\,(T_{12} + T_{21})\,(c(t))\;\dot{r}(t)\;\; \sin r + \dfrac{d}{dt}\left(\left(T_{12} + T_{21}\right)(c(t))\right)\cos r \Big) \|X\| = 0, \\ \Big((T_{12} + T_{21})\,(c(t))\;\dot{r}(t)\;\; \cos r + \dfrac{d}{dt}\left(\left(T_{12} + T_{21}\right)(c(t))\right)\sin r \Big) \|Y\| = 0, \\ T_{22}(c(t))\ddot{r}(t) + T_{11}(c(t))\;\; \left(\|X\|^2 - \|Y\|^2\right)\cos r \sin r + \dot{r}(t)\dfrac{d}{dt}(T_{22}(c(t))) = 0, \end{cases}$$

such that $(\cos r(t), \sin r(t))$ is periodic in t with period T. Then these exhaust all the A-equivariant T-harmonic maps with the condition (3.1) of (T^2, g) into $(\mathbb{S}^n, \mathrm{can})$ with the $(SO(p+1)\times SO(n-p))$–action. In special case $T = fI$, where f is a smooth function on torus and $f(kc(t)) = f(c(t))$, the function $r(t)$ is a solution of ODE

$$f(t)\ddot{r}(t) + f(t)\left(\|X\|^2 - \|Y\|^2\right)\cos r \sin r + \dot{f}(t)\dot{r}(t) = 0.$$

The case f is a non zero constant, we recover [28, equation (3.6)].

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