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HARNACK INEQUALITIES FOR NONLINEAR PARABOLIC EQUATIONS UNDER INTEGRAL RICCI CURVATURE BOUNDS

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Abstract. This work is devoted to the study of gradient estimates and Harnack inequalities for positive solutions to a class of nonlinear parabolic equations on compact Riemannian manifolds under integral Ricci curvature bounds. More precisely, we consider equations of the form:

$$\partial_t v = \Delta v + av(\ln v)^b + q(x,t)A(v),$$

where $a, b \in \mathbb{R}$ and $A(v)$ is a smooth function. These equations naturally generalize the classical heat equation and arise in various geometric and physical contexts. We establish Li-Yau, Hamilton, and Souplet-Zhang type gradient estimates, and deduce corresponding Harnack inequalities. Our results extend previous works by relaxing the curvature conditions to integral lower bounds on the Ricci tensor, thereby allowing for more general geometric settings.

1. Introduction

Understanding the behavior of heat-type equations on Riemannian manifolds has been deeply influenced by gradient estimates, which serve as a bridge between geometric curvature conditions and analytic properties of PDEs. Foundational contributions in this area include the works of Li and

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Yau [13], Hamilton [10], and Zhang and Zhu [23], which have significantly shaped modern geometric analysis.

Let $\rho(y)$ be the minimal eigenvalue of the Ricci curvature operator $\mathrm{Ric} : T_yM \to T_yM$ at a point $y \in M$. Define its negative part as

$$\mathrm{Ric}_-(y) = \max\{-\rho(y), 0\}.$$

Given a point $y \in M$ and radius $r > 0$, the geodesic ball $B(y,r)$ can be used to introduce a scale-sensitive curvature quantity as proposed in [7]:

$$k(y,p,r) = r^2 \left(\int_{B(y,r)} |\mathrm{Ric}_-|^p \, d\mu \right)^{1/p}, \quad k(p,r) = \sup_{y \in M} k(y,p,r).$$

The seminal Li-Yau estimate [13] addresses the heat equation $u_t = \Delta u$ under Ricci lower bounds $\mathrm{Ric} \geq -K$ on complete manifolds, yielding:

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \leq \frac{n\alpha^2 K}{2(\alpha - 1)} + \frac{n\alpha^2}{2t}, \quad \alpha > 1. \tag{1.1}$$

Hamilton [10] later proposed a refinement involving a dynamic weight $\alpha(t) = e^{2Kt}$, resulting in:

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{u_t}{u} \leq e^{4Kt} \frac{n}{2t}.$$

Sun [19] offered another variation where the time-dependence is linear in Kt:

$$\frac{|\nabla u|^2}{u^2} - \left(1 + \frac{2}{3}Kt\right) \frac{u_t}{u} \leq \left(\frac{n}{2t} + \frac{nK}{4}\right)\left(1 + \frac{2}{3}Kt\right) - \frac{nK}{12}.$$

Li and Xu [12] introduced a formulation using hyperbolic functions:

$$\frac{|\nabla u|^2}{u^2} - \left(\frac{\cosh(Kt)\sinh(Kt) - Kt}{\sinh^2(Kt)} + 1\right) \frac{u_t}{u} \leq \frac{n}{2} K(1 + \coth(Kt)).$$

This was further approximated in linear form by [5]:

$$\frac{|\nabla u|^2}{u^2} - \left(1 + \frac{2}{3}Kt\right) \frac{u_t}{u} \leq \frac{n}{2}\left(\frac{1}{t} + K + \frac{1}{3}K^2 t\right).$$

Expanding the scope of gradient estimates, Yang [21] built upon Ma's earlier investigations [14], dealing with equations of the type:

$$u_t = \Delta u + au \log u + bu,$$

on noncompact manifolds with Ricci curvature bounded from below. Additionally, Abolarinwa et al. [1] derived elliptic-type estimates for solutions to:

$$u_t = \Delta_f u - pu^\beta - qu,$$

where $\beta \in \mathbb{R}$ and p, q are smooth functions defined on weighted Riemannian manifolds.

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Further generalizations emerged through the efforts of Zhang and Zhu [23, 24], who obtained Li-Yau-style estimates under the integral curvature constraints introduced by Petersen and Wei [16, 17]:

$$k(p,r) = \sup_{y\in M} r^2 \left(\int_{B(y,r)} |\mathrm{Ric}_-|^p \, d\mu \right)^{1/p} \le \kappa. \tag{1.2}$$

Their results include:

$$\alpha \bar{J}(t) \frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \le \frac{C_1}{\bar{J}} \left(1 + \frac{C_2}{\bar{J}} \right) + \frac{C_3}{t\bar{J}},$$

where $\bar{J}(t)$ is a strictly decreasing positive function, and constants C_i depend on n, p, α. Rose [18] broadened the context to include manifolds where Ric_- belongs to the Kato class. Subsequently, Olive [15] examined Neumann boundary problems within compact domains subject to similar curvature bounds.

2. Main Results

Let (M^n, g) be a compact Riemannian manifold. We study positive solutions $v(x,t)$ of the nonlinear parabolic equation:

$$\partial_t v = \Delta v + av(\ln v)^b + q(x,t)A(v),$$

where $A(v)$ is twice differentiable and $q(x,t)$ is smooth with spatial and temporal derivatives under control. Setting $h = \ln v$, we derive evolution equations for h and construct the auxiliary function

$$G = I|\nabla h|^2 - \alpha h_t + a\alpha h^b + \alpha q B(h) - \alpha\varphi(t),$$

where $B(h)$ encodes the structure of $A(v)$ and φ is a time-dependent function selected to facilitate maximum principle arguments.

Under integral Ricci curvature bounds of the form

$$\left(\int_{B(y,r)} |\,\mathrm{Ric}^-\,|^p \right)^{1/p} \le \kappa r^{-2},$$

and using appropriate cutoff functions, we prove Li-Yau type gradient estimates:

$$\frac{|\nabla v|^2}{v^2} - \alpha \frac{v_t}{v} \le \frac{n\alpha^2}{2(2-\varepsilon)\bar{I}(t)} \left(\frac{1}{t} + C(x,t) \right),$$

where $\bar{I}(t)$ is a decreasing function depending on the geometry, and $C(x,t)$ aggregates the bounds on $A(v)$ and its derivatives, as well as on $q(x,t)$.

By integrating these inequalities along space-time curves, we derive classical Harnack inequalities of the form

$$v(x_1,t_1) \le v(x_2,t_2) \left(\frac{t_2}{t_1} \right)^{\delta} \exp\left(C \int_{t_1}^{t_2} \left(\frac{|\dot{\zeta}(s)|^2}{\bar{I}(s)} + \Psi(s) \right) ds \right),$$

where $\zeta(s)$ is a minimizing path joining x_1 and x_2 in space-time and $\Psi(s)$ involves geometric and analytic terms.

Our estimates include four types:

- Li-Yau type with constant α,
- Hamilton type with $\alpha(t) = \delta e^{\kappa t}$,
- Li-He type with hyperbolic functions,
- Linear Li-He type with $\alpha(t) = \delta + \kappa t$.

These generalizations are new even in the linear case and improve upon known results.

3. Summary of Proofs/Conclusions

The core of the proof involves computing the evolution of the function G under the operator $(\Delta - \partial_t)$ and applying the Bochner-Weitzenböck formula:

$$\frac{1}{2}\Delta|\nabla h|^2 = |\operatorname{Hess} h|^2 + \operatorname{Ric}(\nabla h, \nabla h) + \langle \nabla \Delta h, \nabla h\rangle.$$

We carefully estimate all terms involving $A(v)$, $q(x,t)$, and the curvature, then use the maximum principle with suitable cutoff functions to control the behavior of G.

Integration along space-time paths yields classical Harnack inequalities, confirming that integral curvature bounds are sufficient for obtaining meaningful analytic control on solutions to nonlinear parabolic equations.

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