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SOME SYMMETRIC POLYNOMIAL AND RELATED IDENTITIES

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Abstract. In this paper, we study some symmetric polynomials and their properties. In addition to the elementary and complete symmetric polynomials, we consider the accumulative versions of these polynomials and using common combinatorial tools, particularly generating functions, we study related identities. In order to precise how these identities generalize and extend binomial-type identities, the notation is developed through the paper.

1. Introduction

A polynomial of n variables is called a *symmetric polynomial*, if its value does not change by permuting its variables. Symmetric polynomials appeared naturally in the coefficients of one variable polynomials. Various types of symmetric polynomials have been studied in a wide range of mathematical branches spanning from the old Galois theory [3, 1], combinatorics and algebra [6, 15], representation theory of super Lie algebras [16] to mathematical physics [5] and bioinformatics [14]. The most classical type of symmetric polynomials are elementary symmetric polynomials and complete symmetric polynomials. In this paper, by introducing a new notation, we study these polynomials

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as extensions of binomial type coefficients. Using this new notation we prove new identities for symmetric polynomials which are natural generalization of some well-known binomial identities, such as Vandermond identity.

Here, we give some basic notation and definition which are used in the throughout the paper and some notation which are used in specific part are defined wherever needed. We denote the set of non-negative integers by $\mathbb{N}$. For a positive integer n, we let $[n] = \{1, \ldots, n\}$. For a finite set X the set of all subsets of X is denoted as $P(X)$ and the set of all subsets of X of size i (resp. at most i) is denoted as $P_i(X)$ (respectively $P_{\leq i}(X)$). Also by $P'_i(X)$ (resp. by $P'_{\leq i}(X)$) we mean the set of multi-sets of size i (resp. at most i) whose elements come from X. For a given number n and a given non-negative integer m the falling and rising factorial denoted as $n^{\underline{m}}$ and $n^{\overline{m}}$ are defined as

$$n^{\underline{m}} = n(n-1)\cdots(n-m+1),$$
$$n^{\overline{m}} = n(n+1)\cdots(n+m-1).$$

1.1. **binomial coefficients and some binomial-type numbers.** In this subsection we introduce and fix notation for some types of numbers called binomial-type numbers. Based on the definition of factorial functions we mention definition of four types of numbers, called binomial type numbers. There is nothing new about the classical binomial coefficients; The long history of binomial coefficients and their applications in various fields of mathematics together with their diverse flexible forms leading to magical identities, has made them enough popular to occupy chapters and sections in elementary and advanced textbooks of discrete mathematics. Despite these coefficients, we fix the notation about the other three binomial-type numbers and we mention their combinatorial meanings and a few number of identities. Some of them are useful to show similarities between binomial-type numbers and binomial coefficients and some of them will be generalized to identities of symmetric polynomials in future sections.

Definition 1.1. *Let n be a real number and m be non-negative integers. Then*

$$\binom{n}{m} = \frac{n^{\underline{m}}}{m!},$$
$$\left(\!\!\binom{n}{m}\!\!\right) = \frac{n^{\overline{m}}}{m!},$$
$$\binom{n}{\leq m} = \sum_{i=0}^{m} \binom{n}{i},$$
$$\left(\!\!\binom{n}{\leq m}\!\!\right) = \sum_{i=0}^{m} \left(\!\!\binom{n}{i}\!\!\right).$$

The values of these functions are defined to be 0 if m is not a non-negative integer.

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A wider definition of $\binom{n}{m}$ using the function Γ is given in [2]. However, the above definition which encompasses non-integer (real or complex) values of parameters is still wider than our purpose in the current work. We are mostly interested to non-negative integer parameters. For instance, considering m as a nonnegative integer, the classical combinatorial interpretation of $\binom{n}{m}$ as the number of non-repetitive selection of m objects out of n ones, only covers non-negative integers n; (A similar interpretation of other binomial-type numbers is discussed in Remark 1.4.) An extension of this interpretation to negative integers n is found in [9] and we do not know any combinatorial interpretation for non-integers n.

Remark 1.2. *Let m be a non-negative integer. The value of $\binom{n}{m}$ (resp. $\binom{n}{\leq m}$) equals the number of selection of m objects (resp. at most m objects) out of given n objects. Some results about the order of magnitude of $\binom{n}{\leq m}$ are found in text books (See for instance* [10, Lemma 3.8.2] *and the proof of* [13, Theorem 3.6.1]*). From a geometrical point of view, the value of $\binom{n}{\leq m}$ is number of points inside a closed disc of radius m in the space $\{0,1\}^n$. A generalization of this is mentioned in* [8].

Remark 1.3. *Let m be a non-negative integer. The value of $\left(\!\binom{n}{m}\!\right)$ (resp. $\left(\!\binom{n}{\leq m}\!\right)$) equals the number of selection of m objects (resp. at most m objects) out of n ones with repetitions. It is immediate to see that*

$$\left(\!\binom{n}{m}\!\right) = \binom{n+m-1}{m}.$$

The value of $\left(\!\binom{n}{\leq m}\!\right)$ is given in Remark 1.5.

Remark 1.4. *Let m and n be non-negative integers. Then the following hold:*

(i) $\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$
(ii) $\left(\!\binom{n}{m}\!\right) = \left(\!\binom{n-1}{m}\!\right) + \left(\!\binom{n}{m-1}\!\right)$
(iii) $\binom{n}{\leq m} = \binom{n-1}{\leq m} + \binom{n-1}{\leq m-1}$
(iv) $\left(\!\binom{n}{\leq m}\!\right) = \left(\!\binom{n-1}{\leq m}\!\right) + \left(\!\binom{n}{\leq m-1}\!\right)$

Remark 1.5. *Let m and n be non-negative integers. Then the following hold:*

(i) $\binom{n}{\leq m} = 2^n$, *when $m \geq n$.*
(ii) $\binom{n}{\leq m} + \binom{n}{\leq n-m-1} = 2^n$.
(iii) $\left(\!\binom{n}{\leq m}\!\right) = \left(\!\binom{n+1}{m}\!\right) = \binom{n+m}{m}$.

1.2. **Some symmetric polynomials with a new notation.** The simplest type of symmetric polynomials are elementary symmetric polynomials which appear in college algebra as the coefficients of univariate polynomials. More precisely, if we consider each root of a univariate polynomial $P(x)$ of degree n as an independent variable, then for each i, $1 \leq i \leq n$, the coefficient of x^i is a symmetric polynomial with respect to the roots. This leads to the study of polynomial solutions through the permutation group of the roots, which was the origin of Galois theory [3, 1]. The fundamental theorem of symmetric polynomial states that any symmetric polynomial is expressible in terms elementary symmetric polynomials [4, 11]. Each summand of the ℓ-th elementary polynomial on n variables corresponds to a non-repeated selection of ℓ objects among the n ones; Thus setting the variables to 1, the value of this polynomial equals the good old binomial coefficient $\binom{n}{\ell}$. The next type of these polynomials in the classical part of the theory are complete symmetric polynomials. They admit a similar combinatorial interpretation in terms of the repeated binomial coefficients. It is proved that any elementary symmetric polynomial is expressible in terms of complete symmetric ones, thus as a consequence of the fundamental theorem, any symmetric polynomial is expressible in terms of complete symmetric ones. In this section after mentioning the formal definition of symmetric polynomials and some related notation and identities, we naturally use the notation of binomial coefficients (resp. repeated binomial coefficients) for elementary symmetric polynomials (resp. complete symmetric polynomials). This notation lead us to prove some binomial-type equation for these polynomials.

Definition 1.6. *A polynomial $f \in \mathbb{F}[x_1, x_2, \dots, x_n]$ is called symmetric on n variables if for any permutation σ on $[n]$, $f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$.*

Definition 1.7. *for $\ell \geq 1$, the ℓ-th complete symmetric function in n variables is defined by*

$$h_\ell(x_1, \dots, x_n) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_\ell \leq n} x_{i_1} x_{i_2} \cdots x_{i_\ell},$$

and ℓ-th elementary symmetric polynomial is defined as

$$e_\ell(x_1, \dots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell \leq n} x_{i_1} x_{i_2} \cdots x_{i_\ell}.$$

Definition 1.8. *For a positive integer n we denote an n-tuple $(p_1, \dots, p_n)$ of nonnegative integers by $\overrightarrow{p} = (p_1, \dots, p_n)$. For such an element $\overrightarrow{p} \in \mathbb{N}^n$ we let $||p||_0 = \max(p_1, \dots, p_n)$ and $||p||_1 = p_1 + \cdots + p_n$. We denote the all-one row vector of size n by $\overrightarrow{j_n}$.*

Definition 1.9. *Let $\overrightarrow{x} = (x_1, x_2, \dots, x_n)$ and $\overrightarrow{p} = (p_1, \dots, p_n)$ be n-tuple of variables and n-tuple of integers, respectively. Then we let $\overrightarrow{x}^{\overrightarrow{p}} = x_1^{p_1} \cdots x_n^{p_n}$. Also, for a subset $F = \{f_1, \dots, f_k\}$ of $[n]$, we let $\overrightarrow{x}_F = x_{f_1} \cdots x_{f_k}$.*

Definition 1.10. *For $\overrightarrow{x} = (x_1, \dots, x_n)$ we use notations $\binom{\overrightarrow{x}}{\ell}$ and $\left(\!\binom{\overrightarrow{x}}{\ell}\!\right)$ for $e_\ell(x_1, \dots, x_n)$ and $h_\ell(x_1, \dots, x_n)$ respectively. In other words, we let*

$$\binom{\overrightarrow{x}}{\ell} = \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell \leq n} x_{i_1} x_{i_2} \cdots x_{i_\ell},$$

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$$\left(\!\!\binom{\overrightarrow{x'}}{\ell}\!\!\right) = \sum_{1\le i_1\le i_2\le\cdots\le i_\ell\le n} x_{i_1}x_{i_2}\cdots x_{i_\ell}.$$

We also let $\binom{\overrightarrow{x}}{\le m} = \sum_{i=0}^{m}\binom{\overrightarrow{x}}{i}$ *and* $\left(\!\!\binom{\overrightarrow{x}}{\le m}\!\!\right) = \sum_{i=0}^{m}\left(\!\!\binom{\overrightarrow{x}}{i}\!\!\right)$, *we call these polynomials, accumulated elementary symmetric polynomial and accumulated complete symmetric polynomial, respectively.*

The following proposition is a natural generalization of Pascal's identity.

Proposition 1.11. *Let* $m \ge 0, n > 1$, $\overrightarrow{x} = (x_1,\ldots,x_n)$ *and* $\overrightarrow{x'} = (x_2,\ldots,x_n)$. *Then the following recurrence relations hold.*

(i) $\binom{\overrightarrow{x}}{m} = x_1\binom{\overrightarrow{x'}}{m-1} + \binom{\overrightarrow{x'}}{m}$,

(ii) $\left(\!\!\binom{\overrightarrow{x}}{m}\!\!\right) = x_1\left(\!\!\binom{\overrightarrow{x}}{m-1}\!\!\right) + \left(\!\!\binom{\overrightarrow{x'}}{m}\!\!\right)$.

(iii) $\binom{\overrightarrow{x}}{\le m} = x_1\binom{\overrightarrow{x'}}{\le m-1} + \binom{\overrightarrow{x'}}{\le m}$,

(iv) $\left(\!\!\binom{\overrightarrow{x}}{\le m}\!\!\right) = x_1\left(\!\!\binom{\overrightarrow{x}}{\le m-1}\!\!\right) + \left(\!\!\binom{\overrightarrow{x'}}{\le m}\!\!\right)$.

The following proposition is based on the above notation.

Proposition 1.12. *For every vector* $\overrightarrow{x}$ *and integer* m, *the following equalities hold:*

(i) $\binom{\overrightarrow{x}}{m} = \sum_{I\in P_m([n])}\overrightarrow{x}_I = \sum_{\|\overrightarrow{p}\|_0=1,\|\overrightarrow{p}\|_1=m}\overrightarrow{x}^{\overrightarrow{p}}$

(ii) $\left(\!\!\binom{\overrightarrow{x}}{m}\!\!\right) = \sum_{I\in P'_m([n])}\overrightarrow{x}_I = \sum_{\|\overrightarrow{p}\|_1=m}\overrightarrow{x}^{\overrightarrow{p}}$

iii) $\binom{\overrightarrow{x}}{\le m} = \sum_{I\in P_{\le m}([n])}\overrightarrow{x}_I = \sum_{\|\overrightarrow{p}\|_0=1,\|\overrightarrow{p}\|_1\le m}\overrightarrow{x}^{\overrightarrow{p}}$

(iv) $\left(\!\!\binom{\overrightarrow{x}}{\le m}\!\!\right) = \sum_{I\in P'_{\le m}([n])}\overrightarrow{x}_I = \sum_{\|\overrightarrow{p}\|_1\le m}\overrightarrow{x}^{\overrightarrow{p}}$

Proof. We mention the proof of part (i) for instance. The other parts are proved similarly. Each m-subset of $[n]$ corresponds to an ordered m-tuple of $i_1 < \cdots < i_m$ of integers in $[n]$, hence,

$$\binom{\overrightarrow{x}}{m} = \sum_{I\in P_m([n])}\overrightarrow{x}_I.$$

On the other hand, if we map any $I \in P_m([n])$ to the corresponding characteristic vector $(p_1,\ldots,p_n)$ with $p_i = 1$ if and only if $i \in I$, then we observe that the second summation of part (i) equals the third one. $\square$

Remark 1.13. *If* $\overrightarrow{x} = \overrightarrow{j_n}$, *then the values of* $\binom{\overrightarrow{j_n}}{m}$, $\left(\!\!\binom{\overrightarrow{j_n}}{m}\!\!\right)$, $\binom{\overrightarrow{j_n}}{\le m}$ *and* $\left(\!\!\binom{\overrightarrow{j_n}}{\le m}\!\!\right)$ *equal to* $\binom{n}{m}$, $\left(\!\!\binom{n}{m}\!\!\right)$, $\binom{n}{\le m}$ *and* $\left(\!\!\binom{n}{\le m}\!\!\right)$, *respectively.*

The following remark is a generalization of the first two parts of Remark 1.5, a generalization of the last part of Remark 1.5 would be generalized in part (ii) of Theorem 2.7.

Remark 1.14. *Let $n > 1$, $\overrightarrow{x} = (x_1, \dots, x_n)$ and $\overrightarrow{y} = (\frac{1}{x_1}, \dots, \frac{1}{x_n})$. Then the following recurrence relations hold.*

(i) $\binom{\overrightarrow{x}}{\leq m} = \prod_{i=1}^{n}(1 + x_i)$, *when $m \geq n$.*

(ii) $\binom{\overrightarrow{x}}{\leq m} + x_1 \cdots x_n \binom{\overrightarrow{y}}{\leq n-m-1} = \prod_{i=1}^{n}(1 + x_i)$.

Proposition 1.15. *Let $m \geq 0, n > 1$, $\overrightarrow{x} = (x_1, \dots, x_n)$ and $\overrightarrow{y} = (y_1, \dots, y_n)$. Then the following recurrence relations hold.*

(i) $\binom{\overrightarrow{x}+\overrightarrow{y}}{m} = \sum_{I \in P_{\leq m}([n])} \overrightarrow{x}_I \binom{\overrightarrow{y_{[n]\setminus I}}}{m-|I|} = \sum_{\substack{||\overrightarrow{p}+\overrightarrow{q}||_0 \leq 1, \\ ||\overrightarrow{p}+\overrightarrow{q}||_1 = m}} \overrightarrow{x}^{\overrightarrow{p}} \overrightarrow{y}^{\overrightarrow{q}}$,

(ii) $\left(\!\binom{\overrightarrow{x}+\overrightarrow{y}}{m}\!\right) = \sum_{||\overrightarrow{p}+\overrightarrow{q}||_1 = m} \overrightarrow{x}^{\overrightarrow{p}} \overrightarrow{y}^{\overrightarrow{q}} = \sum_{||\overrightarrow{p}||_1 \leq m} \overrightarrow{x}^{\overrightarrow{p}} \left(\!\binom{\overrightarrow{y}}{m-|\overrightarrow{p}|}\!\right)$,

(iii) $\binom{\overrightarrow{x}+\overrightarrow{y}}{\leq m} = \sum_{||\overrightarrow{p}+\overrightarrow{q}||_0 \leq 1, ||\overrightarrow{p}+\overrightarrow{q}||_1 \leq m} \overrightarrow{x}^{\overrightarrow{p}} \overrightarrow{y}^{\overrightarrow{q}} = \sum_{I \in P_{\leq m}([n])} \overrightarrow{x}_I \binom{\overrightarrow{y_{[n]\setminus I}}}{\leq m-|I|}$,

(iv) $\left(\!\binom{\overrightarrow{x}+\overrightarrow{y}}{\leq m}\!\right) = \sum_{||\overrightarrow{p}+\overrightarrow{q}||_1 \leq m} \overrightarrow{x}^{\overrightarrow{p}} \overrightarrow{y}^{\overrightarrow{q}} = \sum_{||\overrightarrow{p}||_1 \leq m} \overrightarrow{x}^{\overrightarrow{p}} \left(\!\binom{\overrightarrow{y}}{\leq m-|\overrightarrow{p}|}\!\right)$.

Proof. We prove equations of part (i). The other identities are proved similarly. For $i = 1, \dots, n$, let $z_i = x_i + y_i$. Then

$$\begin{aligned} \binom{\overrightarrow{x}+\overrightarrow{y}}{m} &= \sum_{1 \leq j_1 < \cdots < j_m \leq n} z_{j_1} z_{j_2} \cdots z_{j_m} \\ &= \sum_{J \in P_m([n])} \prod_{j \in J} z_j. \end{aligned}$$

On the other hand

$$\begin{aligned} \prod_{j \in J} z_j &= \prod_{j \in J}(x_j + y_j) \\ &= \sum_{I : I \subseteq J} \overrightarrow{x}_I \overrightarrow{y}_{J \setminus I}. \end{aligned}$$

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Hence,

$$\begin{aligned}\binom{\overrightarrow{x}+\overrightarrow{y}}{m} &= \sum_{J\in P_m([n])}\sum_{I:I\subseteq J}\overrightarrow{x}_I\overrightarrow{y}_{J\setminus I}\\
&= \sum_{I\in P_{\leq m}([n])}\sum_{\substack{J:J\supseteq I\\ J\setminus I\in P_{m-|I|}([n]\setminus I)}}\overrightarrow{x}_I\overrightarrow{y}_{J\setminus I}\\
&= \sum_{I\in P_{\leq m}([n])}\overrightarrow{x}_I\sum_{\substack{J:J\supseteq I\\ J\setminus I\in P_{m-|I|}([n]\setminus I)}}\overrightarrow{y}_{J\setminus I}\\
&= \sum_{I\in P_{\leq m}([n])}\overrightarrow{x}_I\binom{\overrightarrow{y_{[n]\setminus I}}}{m-|I|}\end{aligned}\tag{1.1}$$

This proves the first identity to prove the last one, by using Equation (1.1) we obtain

$$\begin{aligned}\binom{\overrightarrow{x}+\overrightarrow{y}}{m} &= \sum_{I\in P_{\leq m}([n])}\sum_{\substack{J:J\supseteq I\\ J\setminus I\in P_{m-|I|}([n]\setminus I)}}\overrightarrow{x}_I\overrightarrow{y}_{J\setminus I}\\
&= \sum_{\substack{\|\overrightarrow{p}+\overrightarrow{q}\|_0\leq 1,\\ \|\overrightarrow{p}+\overrightarrow{q}\|_1=m}}\overrightarrow{x}^{\overrightarrow{p}}\overrightarrow{y}^{\overrightarrow{q}}.\end{aligned}$$

This completes the proof of part (i). □

2. generating functions and combinatorial identities

In this section we use several generating functions and consider their connections to obtain new identities containing symmetric polynomials.

Definition 2.1. *Let* $\overrightarrow{x}=(x_1,\ldots,x_n)$ *and let* $F(t,\overrightarrow{x})$, $G(t,\overrightarrow{x})$, $F^+(t,\overrightarrow{x})$, $F^-(t,\overrightarrow{x})$, $G^+(t,\overrightarrow{x})$ *and* $G^-(t,\overrightarrow{x})$ *be defined as*

$$\begin{aligned}F(t,\overrightarrow{x}) &=(1+tx_1)\cdots(1+tx_n),\\
G(t,\overrightarrow{x}) &=\frac{1}{F(t,\overrightarrow{x})},\\
F^+(t,\overrightarrow{x}) &=\frac{1}{1+t}F(t,\overrightarrow{x}),\\
F^-(t,\overrightarrow{x}) &=\frac{1}{1-t}F(t,\overrightarrow{x}),\\
G^+(t,\overrightarrow{x}) &=\frac{1}{1+t}G(t,\overrightarrow{x}),\\
G^-(t,\overrightarrow{x}) &=\frac{1}{1-t}G(t,\overrightarrow{x}).\end{aligned}$$

Then we have the following proposition.

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Proposition 2.2. *Let* $\overrightarrow{x} = (x_1, \ldots, x_n)$ *and* $n \geq 0$*. Then*

$$\begin{aligned} F(t, \overrightarrow{x}) &= \sum_{i=0}^{n} \binom{\overrightarrow{x}}{i} t^i, \\ G(t, \overrightarrow{x}) &= \sum_{i \geq 0} (-1)^i \left(\!\!\binom{\overrightarrow{x}}{i}\!\!\right) t^i, \\ F^+(t, \overrightarrow{x}) &= \sum_{i \geq 0} (-1)^i \binom{-\overrightarrow{x}}{\leq i} t^i, \\ F^-(t, \overrightarrow{x}) &= \sum_{i \geq 0} \binom{\overrightarrow{x}}{\leq i} t^i, \\ G^+(t, \overrightarrow{x}) &= \sum_{i \geq 0} (-1)^i \left(\!\!\binom{\overrightarrow{x}}{\leq i}\!\!\right) t^i, \\ G^-(t, \overrightarrow{x}) &= \sum_{i \geq 0} \left(\!\!\binom{-\overrightarrow{x}}{\leq i}\!\!\right) t^i. \end{aligned}$$

Definition 2.3. *For a vector* $\overrightarrow{x} = (x_1, \ldots, x_n)$ *and a given number* α*, we let* $\alpha\overrightarrow{x} = (\alpha x_1, \ldots, \alpha x_n)$ *and* $(\alpha, \overrightarrow{x}) = (\alpha, x_1, \ldots, x_n)$*.*

The following proposition states some simple properties of these functions and the proof is straightforward.

Proposition 2.4. *For any number* $\alpha \neq 0$*, functions* $F(t, \overrightarrow{x})$ *and* $G(t, \overrightarrow{x})$ *satisfy the following equations.*

(i) $F(\alpha t, \overrightarrow{x}) = F(t, \alpha\overrightarrow{x}),\ \ G(\alpha t, \overrightarrow{x}) = G(t, \alpha\overrightarrow{x})$.
(ii) $F^+(t, (1, \alpha\overrightarrow{x})) = F^+(\alpha t, (1, \overrightarrow{x})) = F(t, \alpha\overrightarrow{x})$.
(iii) $F^-(t, (-1, \alpha\overrightarrow{x})) = F^-(\alpha t, (-1, \overrightarrow{x})) = F(t, \alpha\overrightarrow{x})$.
(iv) $G^+(t, \alpha\overrightarrow{x}) = G(t, (1, \alpha\overrightarrow{x})), \quad G^+(\alpha t, \overrightarrow{x}) = G(t, \alpha(1, \overrightarrow{x}))$.
(v) $G^-(t, \alpha\overrightarrow{x}) = G(t, (-1, \alpha\overrightarrow{x})), \quad G^-(\alpha t, \overrightarrow{x}) = G(t, \alpha(-1, \overrightarrow{x}))$.

Proof. We prove part (ii) for instance. Using Definition 2.1

$$\begin{aligned} F^+(t, (1, \alpha\overrightarrow{x})) &= \frac{1}{1+t} F(t, (1, \alpha\overrightarrow{x})) \\ &= \frac{1}{1+t} (1+t)(1+t\alpha x_1) \cdots (1+t\alpha x_n) \\ &= \frac{1}{1+\alpha t} (1+\alpha t)(1+\alpha t x_1) \cdots (1+\alpha t x_n) \\ &= F^+(\alpha t, (1, \overrightarrow{x})) = F(t, \alpha\overrightarrow{x}). \end{aligned}$$

□

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If we set $\alpha = 1$ in Proposition 2.4, then we have the following corollary.

Corollary 2.5. *For any integer m and any vector $\overrightarrow{x}$ the following equalities hold:*

(i) $F^{+}(t,(1,\overrightarrow{x})) = F^{-}(t,(-1,\overrightarrow{x})) = F(t,\overrightarrow{x})$.

(ii) $G(t,(1,\overrightarrow{x})) = G^{+}(t,\overrightarrow{x}),\ \ G(t,(-1,\overrightarrow{x})) = G^{-}(t,\overrightarrow{x})$.

Proposition 2.6. *For any number α, any integer m and any vector $\overrightarrow{x}$ the following equalities hold:*

(i) $\binom{\alpha\overrightarrow{x}}{m} = \alpha^m \binom{\overrightarrow{x}}{m}$

(ii) $\left(\!\!\binom{\alpha\overrightarrow{x}}{m}\!\!\right) = \alpha^m \left(\!\!\binom{\overrightarrow{x}}{m}\!\!\right)$

(iii) $\binom{\alpha\overrightarrow{x}}{\leq m} = \alpha^m \binom{\overrightarrow{x}}{\leq m} + (1-\alpha)\sum_{i=0}^{m-1}\alpha^i \binom{\overrightarrow{x}}{\leq i}$

(iv) $\left(\!\!\binom{\alpha\overrightarrow{x}}{\leq m}\!\!\right) = \alpha^m \left(\!\!\binom{\overrightarrow{x}}{\leq m}\!\!\right) + (1-\alpha)\sum_{i=0}^{m-1}\alpha^i \left(\!\!\binom{\overrightarrow{x}}{\leq i}\!\!\right)$

Proof. (i) Using the first identity of Proposition 2.4 (i) we obtain

$$\begin{aligned}\binom{\alpha\overrightarrow{x}}{m} &= [t^m]F(t,\alpha\overrightarrow{x})\\ &= [t^m]F(\alpha t,\overrightarrow{x})\\ &= \alpha^m\binom{\overrightarrow{x}}{m}.\end{aligned}$$

(ii) The proof is similar to that of part (i), using the second identity of Proposition 2.4 (i).

(iii) Using fourth identity of Definition 2.1 and Proposition 2.4 (i)

$$(1-t)F^{-}(t,\alpha\overrightarrow{x}) = (1-\alpha t)F^{-}(\alpha t,\overrightarrow{x}) = F(t,\alpha\overrightarrow{x})$$

Thus

$$F^{-}(t,\alpha\overrightarrow{x}) = (\alpha + \frac{1-\alpha}{1-t})\,F^{-}(\alpha t,\overrightarrow{x})$$

and

$$\begin{aligned}\binom{\alpha\overrightarrow{x}}{\leq m} &= [t^m]F^{-}(t,\alpha\overrightarrow{x})\\ &= \alpha[t^m]F^{-}(\alpha t,\overrightarrow{x}) + (1-\alpha)[t^m]\Big(\frac{1}{1-t}\,F^{-}(\alpha t,\overrightarrow{x})\Big)\end{aligned}$$

Now, it is enough to observe that

$$\begin{aligned}[t^m]\Big(\frac{1}{1-t}\,F^{-}(\alpha t,\overrightarrow{x})\Big) &= \sum_{i=0}^{m}[t^i]F^{-}(\alpha t,\overrightarrow{x})\\ &= \sum_{i=0}^{m}\alpha^i\binom{\overrightarrow{x}}{\leq i}.\end{aligned}$$

(iv) Considering the last identity of Definition 2.1,

$$(1-t)G^{-}(t,\alpha\overrightarrow{x}) = G(t,\alpha\overrightarrow{x})$$
$$(1-\alpha t)G^{-}(\alpha t,\overrightarrow{x}) = G(\alpha t,\overrightarrow{x}).$$

Now, using the second identity of Proposition 2.4 (i),

$$(1-t)G^{-}(t,\alpha\overrightarrow{x}) = (1-\alpha t)G^{-}(\alpha t,\overrightarrow{x}).$$

The rest of the proof is similar to that of part (iii).

□

Theorem 2.7. *For given sequence $\overrightarrow{x}$ and integer m the followings hold*

(i) $\binom{\overrightarrow{x}}{m} = \binom{(-1,\overrightarrow{x})}{\leq m}$.
(ii) $\left(\!\!\binom{\overrightarrow{x}}{\leq m}\!\!\right) = \left(\!\!\binom{(1,\overrightarrow{x})}{m}\!\!\right)$.

Proof. Using Proposition 2.2 and Corollary 2.5, the following equalities are deduced.

(i)

$$\begin{aligned}\binom{\overrightarrow{x}}{m} &= [t^m]F(t,\overrightarrow{x})\\ &= [t^m]F^{-}(t,(-1,\overrightarrow{x}))\\ &= \binom{(-1,\overrightarrow{x})}{\leq m}.\end{aligned}$$

(ii)

$$\begin{aligned}\left(\!\!\binom{(1,\overrightarrow{x})}{m}\!\!\right) &= [t^m]G(t,(-1,-\overrightarrow{x}))\\ &= [t^m]G^{-}(t,-\overrightarrow{x})\\ &= \left(\!\!\binom{\overrightarrow{x}}{\leq m}\!\!\right).\end{aligned}$$

□

Remark 2.8. *Identity given in Theorem 2.7 are generalizations of the following binomial identities, (the first binomial identity is well-known and can be found in* [10, page 51]*, whereas the second identity was mentioned in Remark 1.4).*

(i) $(-1)^m\binom{n}{m} = \sum_{k=0}^{m}(-1)^k\binom{n+1}{k}$.
(ii) $\left(\!\!\binom{n}{\leq m}\!\!\right) = \left(\!\!\binom{n+1}{m}\!\!\right)$.

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To see this it is enough to put $\overrightarrow{x} = -\overrightarrow{j_n}$ in the first identity of Theorem 2.7, and using the identity $\binom{-\overrightarrow{j_n}}{m} = (-1)^m\binom{n}{m}$. And second identity can be deduced by setting $\overrightarrow{x} = \overrightarrow{j_n}$ in the second part of Theorem 2.7 and using identity $\binom{\overrightarrow{j_n}}{m} = \binom{n}{m}$.

In the following examples the validity of Theorem 2.7 is shown, in the case that $\overrightarrow{x}$ is a vector of length 3.

Example 2.9. *Let $\overrightarrow{x} = (x_1, x_2, x_3)$, then*

$$\begin{aligned}\binom{(-1,\overrightarrow{x})}{\leq 2} &= 1 + (-1 + x_1 + x_2 + x_3) + (-x_1 - x_2 - x_3 + x_1x_2 + x_1x_3 + x_2x_3)\\ &= x_1x_2 + x_1x_3 + x_2x_3\\ &= \binom{\overrightarrow{x}}{2},\end{aligned}$$

as it is expected from part (i) of Theorem 2.7.

Example 2.10. *Let $\overrightarrow{x} = (x_1, x_2, x_3)$, then*

$$\begin{aligned}\left(\!\!\binom{(1,\overrightarrow{x})}{2}\!\!\right) &= 1 + x_1^2 + x_2^2 + x_3^2 + x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3\\ &= (x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3) + (x_1 + x_2 + x_3) + 1\\ &= \left(\!\!\binom{\overrightarrow{x}}{2}\!\!\right) + \left(\!\!\binom{\overrightarrow{x}}{1}\!\!\right) + \left(\!\!\binom{\overrightarrow{x}}{0}\!\!\right)\\ &= \left(\!\!\binom{\overrightarrow{x}}{\leq 2}\!\!\right),\end{aligned}$$

as it is expected from part (ii) of Theorem 2.7.

Proposition 2.11. *Let $\overrightarrow{x}$ and $\overrightarrow{y}$ be two vectors of arbitrary length, then the following hold.*

(i) $F(t,\overrightarrow{x})F(t,\overrightarrow{y}) = F(t,\overrightarrow{xy})$,
(ii) $G(t,\overrightarrow{x})G(t,\overrightarrow{y}) = G(t,\overrightarrow{xy})$,
(iii) $F(t,\overrightarrow{x})G(t,\overrightarrow{xy}) = G(t,\overrightarrow{y})$,
(iv) $F(t,\overrightarrow{xy})G(t,\overrightarrow{x}) = F(t,\overrightarrow{y})$,

Proof. These equations are immediately obtained using Definition 2.1. ∎

Theorem 2.12. *Let $\overrightarrow{x}$ and $\overrightarrow{y}$ be two vectors of arbitrary length, then the following hold.*

(i) $\sum_{i=0}^{m}\binom{\overrightarrow{x}}{i}\binom{\overrightarrow{y}}{m-i} = \binom{\overrightarrow{xy}}{m}$,
(ii) $\sum_{i=0}^{m}\left(\!\!\binom{\overrightarrow{x}}{i}\!\!\right)\left(\!\!\binom{\overrightarrow{y}}{m-i}\!\!\right) = \left(\!\!\binom{\overrightarrow{xy}}{m}\!\!\right)$,

(iii) $\sum_{i=0}^{m}(-1)^i\left(\!\!\binom{\overrightarrow{xy}}{i}\!\!\right)\binom{\overrightarrow{x}}{m-i} = (-1)^m\left(\!\!\binom{\overrightarrow{y}}{m}\!\!\right)$,

(iv) $\sum_{i=0}^{m}\binom{\overrightarrow{xy}}{i}\left(\!\!\binom{\overrightarrow{x}}{m-i}\!\!\right) = \binom{\overrightarrow{y}}{m}$.

Proof. Each part of this theorem is easily concluded from the corresponding part of Proposition 2.11. The proof of part (i) for instance is as below

$$\begin{aligned}\binom{\overrightarrow{xy}}{m} &= [t^m]F(t,\overrightarrow{xy}) \\ &= [t^m]F(t,\overrightarrow{x})F(t,\overrightarrow{y}) \\ &= \sum_{i=0}^{m}[t^i]F(t,\overrightarrow{x})[t^{m-i}]F(t,\overrightarrow{y}) \\ &= \sum_{i=0}^{m}\binom{\overrightarrow{x}}{i}\binom{\overrightarrow{y}}{m-i}\end{aligned}$$

□

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