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SOME ASYMPTOTICALLY EUCLIDEAN TANGENT BUNDLES AND THEIR ADM MASSES

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Abstract. Geometric relativity is mostly referred to the study of spatial manifolds in space-time manifolds as main objects in general relativity. Our goal in this paper, is to study spatial models of the universe that are themselves tangent bundles. In Arnowitt, Deser and Misner's formulation of general relativity as a Hamiltonian system, the main quantities that are studied are total energy and mass of a system and in this paper, we will look at the ADM mass for some asymptotically Euclidean tangent bundles. First, we will completely characterize asymptotically Euclidean interpolation metrics on tangent bundles of simple manifolds. We will then consider a family of interpolation metrics that we will call admissible metrics. We will define the notion of lower ADM mass (that is weaker than ADM mass) and estimate the lower ADM mass of admissible metrics. From the said estimate, we show that a stronger version of Schoen and Yau's positive mass and rigidity of positive mass holds for admissible metrics.

1. Introduction

Asymptotically Euclidean Riemannian manifolds are models of isolated universes in general relativity. Indeed the spatial sections of the space-time solving Einstein equations assuming that there is little to no mass at infinity are given by asymptotically Euclidean manifolds.

Keywords: Tangent Bundle, Asymptotically Euclidean Manifold, Arnowitt-Deser-Misner Mass, Geometric Relativity.

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Studying such manifolds and their relativistic mass that is defined by Arnowitt-Deser-Misner [1] (motivated by the notion of mass in Hamiltonian mechanics) are the main topics of study in geometric relativity. Below, we provide an overview of the required materials.

1.1. **Asymptotically Euclidean metrics.** Suppose M is a smooth manifold with dimension $n \geq 3$. A Riemannian structure (M, g) is called asymptotically Euclidean if there exists a bounded set $K \subset M$ such that the complement $M \smallsetminus K$ is the disjoint union of finite number of ends $M_1, \ldots, M_\ell$ such that each end M_k admits coordinates x_i that are given by a diffeomorphism $\Phi_k : M_k \to \mathbb{R}^n \smallsetminus \bar{B}_1(0)$; furthermore, in these end coordinates, the decay asymptotics

$$g_{ij}(x) = \delta_{ij} + O_2(|x|^{-\tau}), \quad \text{as,} \quad |x| \to \infty$$

hold. Here, by $f \in O_2(|x|^{-\tau})$, we mean

$$|f(x)| + |x| \cdot |\partial f(x)| + |x|^2 |\partial^2 f(x)| < C|x|^{-\tau}, \quad \text{when} \quad |x| \to \infty.$$

Schwarzschild Metric as the fundamental example. Perhaps the first and most important example is the Schwarzschild metric. This example is important from the view point of symmetry. In other words, it is the most symmetric solution of the Einstein field equations so we can actually call it a fundamental solution. It is indeed an analog of the “fundamental solution” for the constant scalar curvature equation.

Let us start with a spherically symmetric metric g meaning g is of the warped product form with base $\mathbb{R}$ and fiber $\mathbb{S}^{n-1}$ i.e.

$$g = ds^2 + r(s)^2 g_{\mathbb{S}^{n-1}}$$

for some positive function $r(s)$. Note that the loci where the symmetric spheres are minimal (and indeed totally geodesic) correspond to $r'(s) = 0$.

The Schwarzschild solution of mass m is a rotationally symmetric metric of the form

$$g_{\mathsf{Sch}} := \left(1 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 g_{\mathbb{S}^{n-1}}$$

It is easy to see that the Schwarzschild metric is asymptotically Euclidean with one end and with asymptotic decay rate $\tau = n - 2$.

1.2. **The ADM and lower ADM masses.** Suppose (M^n, g) is an asymptotically Euclidean manifold with ends $M_1, \ldots, M_\ell$. The ADM mass (standing for its originators Arnowitt, Deser and Misner) for each end M_k is given by the formula

$$m_{ADM}\left(M_k, g\right) := \lim_{r \to 0} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \left(\operatorname{div}_0 g - d\left(\operatorname{tr}_0 g\right)\right)(\nu)\, d\mathbb{S}^r$$

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in which, ν is the Euclidean outward normal vector to $\mathbb{S}_r$ and ω_{n-1} denote the volume of standard unit $(n-1)$-sphere. In the end coordinates, this takes the form

$$m_{ADM}\left(M_k, g\right) = \lim_{r\to 0} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \left(g_{ij,i} - g_{ii,j}\right) \left(\frac{x}{|x|}\right) \, d\mathbb{S}^r$$

It is well-known that if $\tau > \frac{n}{2} - 1$, then the ADM mass is a geometric invariant (independent of the chosen end coordinates). For less than this decay rate, multiple masses can arise. The decay rats larger than $n-2$ force the mass to be zero. It is easy to see that the mass of the Schwarzschild metric given above is exactly the parameter m that shows up in its definition.

If the scalar curvature $scal_g$ is integrable, then one can show the limit in the ADM mass exist and is finite.

Here we introduce a relaxed notion of ADM mass and we call it the lower ADM mass. Given a set of coordinates x^i at the infinity of a manifold, it is defined as

$$\underline{m}_{ADM}(M) := \frac{1}{2(n-1)\omega_{n-1}} \varliminf_{r\to\infty} \int_{M\cap S_r} \left(g_{ij,i} - g_{ii,j}\right) \nu^j dS_r.$$

with values in extended real numbers, so the lower ADM masses $\pm\infty$ are allowed. The benefit of this relaxed notion of mass is that it can be applied to more general manifolds and only requires a set of coordinates at infinity.

Positive Mass Theorem. So it is natural to expect the right definition of mass yield a positive mass in non-trivial universes and if the mass vanishes the space must be vaccum. This was known as the positive mass conjecture in general relativity.

Perhaps the most important result in this direction is the Schoen and Yau's proof of this conjecture in the case which is most relevant from the point of view of classical physics i.e. in three dimensions; this can be found in [8, 10]. This proof soon after was generalized by Schoen and Yau to dimensions less than 8 in [9]. Their proof essentially relies on the regularity of minimal submanifolds which holds in dimensions less than 8.

Theorem 1.1 (Positive Mass Theorem)**.** *Let (M, g) be a complete asymptotically Euclidean manifold with decay rate $\tau < \frac{n}{2} - 1$ and with integrable scalar curvature. Then the ADM mass of each end of M is a nonnegative quantity.*

Theorem 1.2 (Positive Mass Rigidity)**.** *Let (M, g) be a complete asymptotically Euclidean manifold with decay rate $\tau < \frac{n}{2} - 1$ and with integrable scalar curvature. Then vanishing of the ADM mass of any of the ends of M implies (M, g) is isometric to the Euclidean space $\mathbb{R}^n$.*

Following the proof of Schoen and Yau, Witten in [12] presented a proof that works in all dimensions however it requires the existence of a spin structure.

In recent years, there have been other attempts at proving the positive mass theorem and its rigidity in all dimensions as presented in [6, 7] and in [11].

In a totally different directions and using different techniques, Hein and Le Brun [3] have related the ADM mass of an asymptotically Euclidean Kählerian manifold to its topological invariants and have succeeded in proving a positive mass theorem in this setting as well.

1.3. **Mass of tangent bundles of asymptotically Euclidean manifolds.** As was mentioned there are studies in the literature on the ADM mas of asymptotically flat Kählerian manifolds. So it is also natural to consider the much harder case of ones with only an almost complex structures or other weaker than Kählerian cases.

A quintessential example of manifolds admitting almost complex structures are tangent bundles. So this brings us to to question of "what can be said about topology, geometry and the ADM mass of asymptotically Euclidean tangent bundles". In the previous work of the author with collaborators, the topological classification of asymptotically Euclidean tangent bundles is given [2].

From the geometrical and analytical point of view, one can consider tangent bundles and study their masses in relation to the mass of the underlying manifold (when the latter makes sense). In these notes, we wish to pioneer such a study by proving a version of positive mass theorem and rigidity for a special class of asymptotically Euclidean Riemannian metrics on tangent bundles of simple manifolds; metrics that we call admissible.

First we note the following important topological classification result: TM has one asymptotically Euclidean end if and only if M is contractible [2] thus the tangent bundle must be $\mathbb{R}^{2n}$. So to be able to consider more general cases, we will need to allow asymptotically Euclidean manifolds whose tangent bundles are no longer asymptotically Euclidean in the usual sense. However if we assume M is simple (i.e. is an open subset of $\mathbb{R}^n$ or equivalently describable by a single coordinate chart), then TM is also simple (by the parallelizability of open subsets of $\mathbb{R}^n$) and inherits a standard set of coordinates from M.

For the said reason, in these notes, we will only consider simple manifolds M that are asymptotically Euclidean with one end and will look at the lower ADM mass of the tangent bundle TM equipped with a metric $\tilde{g}$ that in the inherited coordinates decays to the Euclidean metric (hence is asymptotically Euclidean in a generalized sense).

2. Main Results

Given a fixed arbitrary Riemannian metric $\hat{h}$ on M with Levi-Civita connection $\widehat{\nabla}$, one can express any other Riemannian metric $\tilde{g}$ on TM via

$$\begin{cases} \tilde{g}_{(p,u)}\left(X^h, Y^h\right) = g^h_{(p,u)}(X,Y) \\ \tilde{g}_{(p,u)}\left(X^h, Y^v\right) = a^c_{(p,u)}(X,Y) \\ \tilde{g}_{(p,u)}\left(X^v, Y^v\right) = g^v_{(p,u)}(X,Y) \end{cases}, \tag{2.1}$$

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where X^v and X^h respectively denote the vertical and horizontal lifts of a tangent vector X to TTM at the point $(p,u) \in TM$. Here $g^h_{(\cdot,u)}$ and $g^v_{(\cdot,u)}$ are families of Riemannian metrics and $a^c_{(\cdot,u)}$ a family of symmetric 2-tensors, all parametrized by u; see [4] for further details.

We define the $\hat{h}$-interpolating metrics as ones of the following form

$$\begin{cases} \tilde{g}_{(p,u)}\left(X^h, Y^h\right) = \left(\varphi_1(u) g_p + (1-\varphi_1(u))\delta\right)(X,Y) \\ \tilde{g}_{(p,u)}\left(X^h, Y^v\right) = 0 \\ \tilde{g}_{(p,u)}\left(X^v, Y^v\right) = \left(\varphi_2(u) g_p + (1-\varphi_2(u))\delta\right)(X,Y) \end{cases} , \varphi_i : \mathbb{R}^n \to \mathbb{R}$$

with the asymptotics

$$0 \le \varphi_i \le 1, \quad \varphi_i \in O_2(|u|^{-\tau_i}), \quad \tau_1 > 0, \quad \tau_2 > 1, \quad \varphi_i(0) = 1, \quad \lim_{|u| \to \infty} \varphi_i(u) = 0.$$

If $\hat{h}$ is a flat metric, we will call this a δ-interpolating metric; in local coordinates, δ-interpolating metrics take the form

$$\begin{cases} \tilde{g}_{ij} = \varphi_1 g_{ij} + (1-\varphi_1)\delta_{ij} \\ \tilde{g}_{i\bar{j}} = 0 \\ \tilde{g}_{\bar{i}\bar{j}} = \varphi_2 g_{ij} + (1-\varphi_2)\delta_{ij} \end{cases} . \tag{2.2}$$

Finally, admissible metrics are defined to be δ-interpolating metrics where we further assume

- g is a complete Riemaanian metric and is conformally flat i.e. $g = U^{\frac{4}{n-2}}\delta$ for some smooth $U > 0$;
- φ_1 has compact support;
- $\varphi_{2,s} \le 0$ (φ_2 is nonincreasing in radial directions);
- $(\mathrm{tr}_0\, g)_r \le 0$ (trace of g with respect to the Euclidean metric is nonincreasing in radial directions).

We will be concerned with simple base manifolds M that means they are described by one coordinate chart (open subsets of $\mathbb{R}^n$). We will also use the following notion of lower ADM mass

$$\underline{m}_{ADM}(M) := \frac{1}{2(n-1)\omega_{n-1}} \varliminf_{r \to \infty} \int_{M \cap S_r} \left(g_{ij,i} - g_{ii,j}\right) \nu^j dS_r.$$

For a rather exhaustive theory of geometric relativity, we refer the reader to [5].

Our first result is the characterization of asymptotically Euclidean $\hat{h}$-interpolating metrics on the tangent bundle of an asymptotically Euclidean manifold as follows:

Theorem 2.1. *An $\hat{h}$-interpolating metric $\tilde{g}$ on the tangent bundle of a simple manifold M is asymptotically Euclidean if and only if $\hat{h}$ is a flat metric (i.e. $\tilde{g}$ is a δ-interpolating metric).*

Our second main result is the following version of positive mass theorem and rigidity of positive mass for admissible metrics.

http://dx.doi.org/10.22108/msci.2025.143588.1713

Theorem 2.2. *Suppose M is a simple manifold with a single end and suppose $\tilde{g}$ is an admissible metric on TM.*

(1) *if $scal_g \geq 0$, then $(TM, \tilde{g})$ has non-negative (possibly $+\infty$) lower ADM mass;*
(2) *If (M, g) is asymptotically Euclidean with decay rate $\tau > \frac{n-2}{2}$, then $\underline{m}_{ADM}(\tilde{g}) = 0$ if and only if g and $\tilde{g}$ are flat metrics and $M \equiv \mathbb{R}^n$ and $TM \equiv \mathbb{R}^{2n}$*

3. Summary of Proofs/Conclusions

3.1. Summary of Proof of Theorem 2.1.

Recalling

$$\partial_i^h = \partial_i - u^b \widehat{\Gamma}_{ib}^k \partial_{\bar{k}} = \partial_i - \lambda_{ik}\partial_{\bar{k}}, \quad \partial_i^v = \partial_{\bar{i}},$$

where $\lambda_{ik} := u^b \widehat{\Gamma}_{ib}^k$, one can compute the metric coefficients as

$$\begin{aligned}
\tilde{g}_{ij} &= \tilde{g}\left(\partial_i^h + \lambda_{ik}\partial_k^v, \partial_j^h + \lambda_{jk}\partial_k^v\right) \\
&= \varphi_1 g_{ij} + (1-\varphi_1)\delta_{ij} + \lambda_{ik}\lambda_{jm}\left(\varphi_2 g_{km} + (1-\varphi_2)\delta_{km}\right) \\
&= \varphi_1 g_{ij} + (1-\varphi_1)\delta_{ij} + \varphi_2\lambda_{ik}\lambda_{jm}g_{km} + (1-\varphi_2)\lambda_{ik}\lambda_{jk},
\end{aligned}$$

$$\begin{aligned}
\tilde{g}_{i\bar{j}} &= \tilde{g}\left(\partial_i^h + \lambda_{ik}\partial_k^v, \partial_j^v\right) \\
&= \lambda_{ik}\left(\varphi_2 g_{kj} + (1-\varphi_2)\delta_{kj}\right) \\
&= \varphi_2\lambda_{ik}g_{kj} + (1-\varphi_2)\lambda_{ij},
\end{aligned}$$

$$\tilde{g}_{\bar{i}\bar{j}} = \tilde{g}\left(\partial_i^v, \partial_j^v\right) = \varphi_2 g_{ij} + (1-\varphi_2)\delta_{ij}.$$

therefore,

$$\tilde{g}_{ij} - \delta_{ij} = \varphi_1(g_{ij} - \delta_{ij}) + \varphi_2\lambda_{ik}\lambda_{jm}g_{km} + (1-\varphi_2)\lambda_{ik}\lambda_{jk},$$

$$\tilde{g}_{i\bar{j}} - \delta_{i\bar{j}} = \tilde{g}_{i\bar{j}} = \varphi_2\lambda_{ik}g_{kj} + (1-\varphi_2)\lambda_{ij},$$

$$\tilde{g}_{\bar{i}\bar{j}} - \delta_{\bar{i}\bar{j}} = \varphi_2(g_{ij} - \delta_{ij}).$$

in particular one can show

$$\tilde{g}_{i\bar{j}} = \varphi_2\lambda_{ik}(g_{kj} - \delta_{kj}) + \lambda_{ij}. \tag{3.1}$$

If $\hat{h}$ is flat then it is easy to see the "if" part holds using standard computations. For the "only if" part, the decay assumptions lead to

$$\lim_{|u|\to\infty} \lambda_{ij}(x,u) = 0, \quad \forall x \in M,$$

http://dx.doi.org/10.22108/msci.2025.143588.1713

where upon substituting $u^b = t\delta_{rb}$ gives us

$$0 = \lim_{|u|\to\infty} \lambda_{ij} = \lim_{|u|\to\infty} u^b \hat{\Gamma}^j_{ib} = \lim_{t\to\infty} t\hat{\Gamma}^j_{ir}(x);$$

and as a result,

$$\hat{\Gamma}^j_{ir}(x) = 0, \quad \forall x \in M.$$

3.2. **Summary of Proof of Theorem 2.2.** Set

$\rho^2 = r^2 + s^2$ and let ν_x and ν_u denote the outward normal vectors in base and in fibers. As is usual the Greek index α runs over both values i corresponding to space and values $\bar{i}$ that correspond to the coordinates on tangent fibers. Starting from

$$\begin{cases} \tilde{g}_{(p,u)}\left(X^h, Y^h\right) = \left(\varphi_1(u)U^{\frac{4}{n-2}} + (1-\varphi_1(u))\right)\delta(X,Y) \\ \tilde{g}_{(p,u)}\left(X^h, Y^v\right) = 0 \\ \tilde{g}_{(p,u)}\left(X^v, Y^v\right) = \left(\varphi_2(u)U^{\frac{4}{n-2}} + (1-\varphi_2(u))\right)\delta(X,Y) \end{cases},$$

set

$$\Phi_1 := \left(\varphi_1 U^{\frac{4}{n-2}} + (1-\varphi_1)\right)^{\frac{n-2}{4}}, \quad \Phi_2 := \left(\varphi_2 U^{\frac{4}{n-2}} + (1-\varphi_2)\right)^{\frac{n-2}{4}}.$$

For any given metric g, let I_g denote the integrand in the ADM mass formula.

In the conformal case, we have

$$I_g = -\frac{4(n-1)}{n-2} U^{\frac{6-n}{n-2}} U_r.$$

Using

$$\nu^j = \frac{x^j}{\rho} = \frac{r}{\rho}\frac{x^j}{r} = \frac{r}{\rho}\nu^j_x, \quad \nu^{\bar{j}} = \frac{u^j}{\rho} = \frac{s}{\rho}\frac{u^j}{s} = \frac{s}{\rho}\nu^j_u$$

With some detailed computations we get to

$$I_{\tilde{g}} = I + \bar{I}$$

where

$$\begin{aligned} I := \left(\tilde{g}_{\alpha j,\alpha} - \tilde{g}_{\alpha\alpha,j}\right)\nu^j &= -\frac{r}{\rho}\frac{4(n-1)}{n-2}\Phi_1^{\frac{6-n}{n-2}}\Phi_{1,r} + \left(\tilde{g}_{\bar{i}j,\bar{i}} - \tilde{g}_{\bar{i}\bar{i},j}\right)\nu^j \\ &= -\frac{r}{\rho}\frac{4(n-1)}{n-2}\Phi_1^{\frac{6-n}{n-2}}\Phi_{1,r} - \varphi_2 g_{ii,j}\nu^j \\ &= -\frac{r}{\rho}\frac{4(n-1)}{n-2}\Phi_1^{\frac{6-n}{n-2}}\Phi_{1,r} - \frac{r}{\rho}\varphi_2(\mathrm{tr}_0 g)_r \\ &= \frac{r}{\rho}\varphi_1 I_g - \frac{r}{\rho}\varphi_2(\mathrm{tr}_0 g)_r. \end{aligned}$$

and

$$\begin{aligned}\bar{I} := \left(\tilde{g}_{\alpha\bar{j},\alpha} - \tilde{g}_{\alpha\alpha,\bar{j}}\right)\nu^{\bar{j}} &= -\frac{s}{\rho}\frac{4(2n-1)}{2n-2}\Phi_2^{\frac{6-n}{n-2}}\Phi_{2,s} - \tilde{g}_{ii,\bar{j}}\nu^{\bar{j}} \\ &= -\frac{s}{\rho}\frac{4(n-1)}{n-2}\Phi_2^{\frac{6-n}{n-2}}\Phi_{2,s} - \varphi_{1\bar{j}}(g_{ii}-\delta_{ii})\nu^{\bar{j}} \\ &= -\frac{s}{\rho}\frac{4(n-1)}{n-2}\Phi_2^{\frac{6-n}{n-2}}\Phi_{2,s} - \frac{s}{\rho}(\mathrm{tr}_0 g - n)\varphi_{1,s} \\ &= -(n-1)\frac{u}{\rho}\left(U^{\frac{4}{n-2}}-1\right)\varphi_{2,s} - \frac{u}{\rho}(\mathrm{tr}_0 g - n)\varphi_{1,s}.\end{aligned}$$

Then admissiblity implies

$$I \geq \frac{r}{\rho}\varphi_1 I_g, \quad \bar{I} \geq 0;$$

The last part of the proof consists of using the co-area formula for the function $F : (x,u) \to \|u\|$ which along with some further analysis leads to the computations

Using the co-area formula we find a lower estimate for the lower ADM mass of $\tilde{g}$.

$$\begin{aligned}&\int_{TM\cap S_\rho} I_{\tilde{g}} dS_\rho \geq \int_0^\rho \int_{A_s^\rho} \frac{\sqrt{\rho^2-s^2}}{\rho}\int_{S_{\sqrt{\rho^2-s^2}}} \varphi_1(u) I_g(x)\, dS_{\sqrt{\rho^2-s^2}}\, dS_s\, ds \\ &= \int_0^R \int_{A_s^\rho} \frac{\sqrt{\rho^2-s^2}}{\rho}\varphi_1(u)\int_{S_{\sqrt{\rho^2-s^2}}} I_g(x)\, dS_{\sqrt{\rho^2-s^2}}\, dS_s\, ds,\end{aligned}$$

where

$$A_s^\rho := \left\{u \in S_s \;:\; \{u\}\times S_{\sqrt{\rho^2-s^2}} \subset TM \cap S_\rho\right\},$$

and R is chosen so that $\mathrm{supp}(\varphi_1) \subset B_R(0)$.

One can see that for any $s \leq R$ and for $\rho >> 1$, we have

$$S_{\sqrt{\rho^2-u^2}} \subset M$$

Also one verifies the asymptotic estimates

$$\left|\varphi_1(u)2(n-1)\omega_{n-1}m - \varphi_1(u)\int_{S_{\sqrt{\rho^2-s^2}}} I_g(x)\, dS_{\sqrt{\rho^2-s^2}}\right| < \epsilon,$$

and

$$\left|\frac{\sqrt{\rho^2-s^2}}{\rho} - 1\right| < \epsilon$$

for $s \leq R$ and ε sufficiently small and ρ sufficiently large.

Hence, we have

$$\underline{\lim}_{\rho\to\infty}\int_{S_{\sqrt{\rho^2-s^2}}} I_g(x)\, dS_{\sqrt{\rho^2-s^2}} = \underline{\lim}_{r\to\infty}\int_{S_r} I_g(x)\, dS_r = 2(n-1)\omega_{n-1}m_{ADM}(g). \tag{3.2}$$

http://dx.doi.org/10.22108/msci.2025.143588.1713

Now using the above equation and using Fatou's lemma twice we can compute as follows

$$\begin{aligned}
\underline{m}_{ADM}(\tilde{g}) &= \varliminf_{\rho\to\infty} \int_0^R \int_{A_s^\rho} \frac{\sqrt{\rho^2 - s^2}}{\rho} \varphi_1(u) \int_{S_{\sqrt{\rho^2-s^2}}} I_g(x)\, dS_{\sqrt{\rho^2-s^2}}\, dS_s\, ds \\
&\geq \int_0^R \varliminf_{\rho\to\infty} \int_{A_s^\rho} \frac{\sqrt{\rho^2 - s^2}}{\rho} \varphi_1(u) \int_{S_{\sqrt{\rho^2-s^2}}} I_g(x)\, dS_{\sqrt{\rho^2-s^2}}\, dS_s\, ds \\
&\geq \int_0^R \int_{S_s} \varliminf_{\rho\to\infty} \frac{\sqrt{\rho^2 - s^2}}{\rho} \varphi_1(u) \int_{S_{\sqrt{\rho^2-s^2}}} I_g(x)\, dS_{\sqrt{\rho^2-s^2}}\, dS_s\, ds \\
&= \int_0^R \int_{S_s} \varphi_1(u) \varliminf_{\rho\to\infty} \int_{S_{\sqrt{\rho^2-s^2}}} I_g(x)\, dS_{\sqrt{\rho^2-s^2}}\, dS_s\, ds \\
&= \int_0^R \int_{S_s} \varphi_1(u) 2(n-1)\omega_{n-1} m_{ADM}(g)\, dS_s\, ds \\
&= 2(n-1)\omega_{n-1} m_{ADM}(g) \int_0^R \int_{S_s} \varphi_1(u)\, dS_s\, ds \\
&= 2(n-1)\omega_{n-1} m_{ADM}(g) \int_{B_R(0)} \varphi_1(u)\, du.
\end{aligned}$$

This key inequality in conjunction with the classical positive mass theorem applied to (M, g) gives the conclusion.

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