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OF CHEESE AND CRUST: A PROOF OF THE PIZZA CONJECTURE AND OTHER TASTY RESULTS**

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Abstract. [Translator's abstract]: For a given natural number N, a circular pizza is divided into $2N$ equiangular slices by means of N straight, concurrent cuts at an arbitrary point P of the interior of the pizza. Then, each slice of the pizza is shared by two individuals ("Gray" and "White"), who alternate slices. A natural question that may arise is that does the area of the gray slices exceed that of white slices? When N is even, an answer to this question has already been given. In this paper, whenever N is an odd number, a complete answer to this problem is given. Moreover, the method of proof are generalized to three dimensional pizzas, so called "calzones", that are some type of pizzas which are obtained by filling the space above the circular pizzas with cheese surrounded by means of upper surfaces such as paraboloid, semi-ellipsoid or cone, and both volumes and surface areas of the Gray and White slices are computed.

1. Introduction

A geometric dissection is a cutting of one or more geometric figures into finitely many pieces that can be rearranged to form one or more other geometric figures. As visual demonstration of relationships

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such as the Pythagorean theorem, dissections have had a surprisingly rich history, reaching back to Greek mathematicians more than two millennia ago. Other examples of mathematical relationships illustrated by dissections include identities of sums of squares and sums of cubes of natural numbers.

More published dissections are either polygons or polyhedra. However there are a few dissections of curved figures. They are challenging to create, because a convex curved boundary in an original figure persists in a target figure unless there is a matching concave curved boundary in the original figure to offset it. It is not easy to identify curved figures that have this property and also interesting objects for dissection.

A remarkable example arises when the pieces are created from a circular disk that is divided by chords, all of which pass through a single point within the disk equally spaced angles. This sort of partition is typical of the way circular pizzas are cut, recognizing that whoever cuts the pizzas may not be careful to have the intersection of the chords be at the center of the disk.

When this chordal intersection point is at the center of the disk, then all resulting slices are congruent. However, when this point does not coincide with the center, then no two slices of opposite color are congruent, unless one of the chords is a diameter. So a natural and intriguing question may arises:

Question: For an arbitrary natural number N, a circular pizza is divided into $2N$ equiangular slices by means of N straight, concurrent cuts at an arbitrary point P of the interior of the pizza. Then, each slice of the pizza is shared by two individuals ("Gray" and "White"), who alternate slices. When does the total area of the gray slices exceeds that of white slices?

When $N = 1$ the situation is completely clear. When $N = 2$, the result seems pretty obvious, but a "proof with few words" may be given. A half century ago, Leslie J. Upton, of suburban Toronto in Canada, observed a curious fact that holds even when two slices are congruent [16]: His observation was based on the fact that if four lines in a plane are concurrent at a point P so that the angle between the lines are each $45°$. Then a circle is superimposed on the configuration of lines so that P lies within the circles. Then he asked to show that the alternate sectors cover one half of the circle. Moreover he asked to show this result without use of calculus.

Michael Goldberg was the first person who gave a positive answer to this question and observed that whenever there are $N \geq 4$ chords and N is even, then the total area of one set of slices equals the total area of the other [6]. By this observation, he was able to give a proof for even $N \geq 4$. This is the so-called "cheese pizza theorem".

In [4] the case for $N = 4$ is illustrated by Carter and Wagon as a "proof without word". Moreover, Larry Carter and Stan Wagon found a nice dissection proof for Upton's original case of four chords [3]. They cut the four slices in one set into eight pieces that assemble to make the four slices in the other set.

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The results for even N are sprinkled throughout the literature, and seem to make their first appearance with a solution by Goldberg [6] (see also [7], [8], [13], [14], [1], [2], and [9]). Proofs for even $N \geq 4$ are accompished by calculating exact area of certain sets of opposite sclices.

For odd N, the total areas of gray slices and the white slices are shown to be unequal by an elegant continuity argument of Don Coppersmith (see [4]), utilizing the transcendence of π. However Coppersmith's proof does not tell us which total area is greater. In [3], Carter, Duncan, and Wagon proposed the problem for the cases $N = 3$ and 5. Though it is not stated there, they made the general conjecture for all odd ≥ 3, observing that the location of the center alone determines which set of slices has greater area. Stanley Rabinowitz mentions this in exercise 26.3 of his as-yet-unpublished manuscript [15] which contains a host of related resultsand it is also stated in [11] (a book of intriguing mathematical mysteries), in the section titled "The Pizza Problem".

The contribution of this paper to "cheese pizza theorem" will be to settle the case for odd $N \geq 3$– the "pizza conjecture" (as it is known in some circles).

2. Main Results

The main result of this paper is the following theorem, which provides a precise solution to the above question, had its genesis in a problem posed by L. J. Upton in Mathematics Magazine almost fifty years ago.

Theorem 2.1. *[Cheese Pizza Theorem.] For a positive integer N, divide a pizza into $2N$ slices by choosing an arbitrary points P in the pizza and making N straight cuts through P, the cuts meeting to form $2N$ equal angles. Alternatively color adjacent slices gray and white. Let O denote the center of the pizza. The total areas of gray and white will then satisfy the following.*

(a) *When $N \geq 4$ is even, the gray area equals the white, but for all N, gray equals white if and only if O lies on a cut.*
(b) *If O does not lie on a cut and $N = 1$, $N = 2$ or N is odd with $N \equiv 3 (\mod 4)$, then gray exceeds white if and only if O lies in a gray slice.*
(c) *If O does not lie on a cut and N is odd with $N \geq 5$ and $N \equiv 1 (\mod 4)$, then white exceeds gray if and only if O lies in a gray slice.*

This implies if the disk is a uniformly distributed cheese pizza shared by gray and white, then for even $N \geq 4$, gray and white get equal amounts. When O lies on a cut, gray and white share equally by symmetry. Therefore, we will henceforth *assume that O does not lie on a cut, but lie within a gray slice*, as in Figure 1. In that case, when $N = 2$ or is odd, gray will never equal white, and we will know which set of slices to choose to get the greater amount of cheese: if $N = 2, 3, 7, 11, \ldots$, then gray will get more; if $N = 5, 9, 13, \ldots$, then white will win.

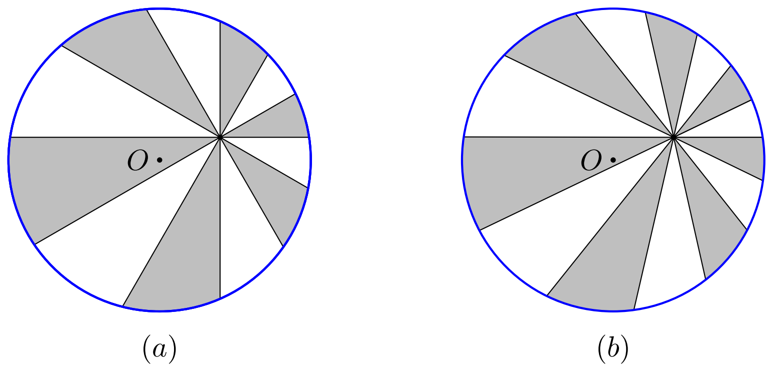

FIGURE 1. Two examples of equiangular sliced pizzas. (a) Gray area equals white when $N > 2$ is even. (Here, $N = 6$.) (b) Gray and white areas differ when N is odd. (Here, $N = 7$.)

3. **Summary of Proofs/Conclusions**

Our proof of pizza conjecture will amount to calculating gray minus white and will consist of several parts. First we show that differences of areas of opposite slices can be expressed in terms of two “central strips” and such strips have an easy area formula we call a “strip function” $s(r)$, where r is the thickness of the strip. Using these strips we then find a fairly simple expression for the difference between the total areas of gray and white slices, reducing the pizza problem to an inequality having an alternative sum involving $s(r)$. Fortunately, $s(r)$ has such a convenient power series expansion that we are able to further reduce the problem to solving the inequality when the function $s(r)$ is replaced by the function r^n, where $n \geq 3$ is odd. Unfortunately, the argument of s take the form of a sum of sine functions, so we have to deal with binomial expansion of these sums, leading to complicated sums of powers of sines. The good news is that applying standard manipulations (involving complex exponential functions and a geometric sum of roots of unity) to those powers of sines yields a sum that contains a product of two alternating lacunary binomial sums. But the *great* news is that these alternating sums turn out to represent the number of certain lattice paths in the plane and hence are never negative. So at the end we travel along lattice paths to deliver the pizza. And it gets even better, because the methods here that solve the pizza problem also *make it possible to handle some specialty menu items-similar divisions of hemispheres, cones, and other figures.* by filling the space above the circular pizza with cheese surrounded by upper surface such as parabolid, semi-ellipsoid or cone, so called “calzone”, slicing them as in the cheese pizza theorem and computing both volumes and surface area of the gray and white slices. Further modifications include both thick crust and thin crust pizza and calzone.

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