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CALABI FLOW ON RIEMANN SURFACES

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Abstract. An important theorem in complex analysis is the uniformization theorem. As a result of uniformizationn theorem, any compact Riemann surface admits a metric of constant Gaussian curvature. A modern way to prove the uniformization theorem is to use geometric flows. Since seminal work of Perleman, many authors prove it using curvature flows. In an important work, Tian used Ricci flow to prove the uniformization theorem. Later Chen used the Calabi flow to give another proof of the Theorem. In contrast to Ricci flow, Calabi flow is a fourth order parabolic PDE. It is quite difficult to deal with fourth order PDEs partially due to lack of any maximum principle. Even proving the long time existence of Calabi flow is hard and it is still open in complex dimension $n \geq 2$. In a breakthrough, Chen-Cheng provide a strong tool in order to deal with certain fourth order nonlinear PDEs. In this article, appealing to the results of Chen-Cheng, we give a different proof for the long time existence of the Calabi flow on compact Riemann surfaces of positive genus.

1. Introduction

A classical theorem in complex analysis is the Uniformization theorem. A consequence of the uniformization theorem, any compact Riemann surface admits a metric of constant Gaussian curvature. From PDE point of view, one needs to solve some elliptic semi linear equations. It is not hard to do so since there has been an extensive literature on the subject.

Since the celebrated work of Perleman on Ricci flow, there have been several proofs of uniformization theorem using geometric flows. Tian proved the uniformization using Ricci flow. Chen proved it using Calabi flow.

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In this note, we introduce the Calabi flow and show the long time existence of it on compact Riemann surfaces of positive genus.

2. Main Results

Let M be a compact Riemann surface and $\tilde{h}$ be a fixed Riemannian metric on M. For simplicity, we assume that $\mathrm{Vol}(M,\tilde{h})=1$.

Let $h_t=e^{u_t}\tilde{h}$ be a smooth family of Riemannian metrics on M. We say, h_t is a Calabi flow with initial condition $h_0=e^{u_0}\tilde{h}$ if

$$\frac{du_t}{dt}=\Delta_{h_t}K_{h_t}.$$

Equivalently,

$$\frac{dh_t}{dt}=\Big(\Delta_{h_t}K_{h_t}\Big)h_t.$$

It is a standard result in parabolic PDEs that for any given initial data h_0, Calabi flow starting from h_0 exists for short time. More precisely, For any given h_0, there exists a positive number t_0 and Riemannian metrics $h_t=e^{u_t}\tilde{h}$ for $t\in[0,t_0)$ such that h_t is a Calabi flow with initial condition $h_0=e^{u_0}\tilde{h}$. The long time existence is more involved. The main result of this paper is the following:

Let M be a Riemann surface of positive genus. For any given h_0, there exist Riemannian metrics $h_t=e^{u_t}\tilde{h}$ for $t\in[0,\infty)$ such that h_t is a Calabi flow with initial condition $h_0=e^{u_0}\tilde{h}$.

The above theorem was first proved by Chruściel for any genus. Later Chen and Struwe gave alternative proof using Mabuchi and Calabi functionals respectively. The goal of this note is to give a different proof in the positive genus case based on the recent work of Chen-Cheng.

3. Summary of Proofs

An essential tool to study geometric flows are quantities that are monotone along the flow. In the case of Calabi flow, the Calabi functional plays a crucial role.

For any metric $he^u\tilde{h}$ on M, we can define the Calabi energy and antropy as follows:

$$\mathrm{Ca}(h)=\int_M K_h^2 d\mu_h,$$

$$\mathrm{Ent}(h)=\int_M ue^u d\mu_{\tilde{h}}=\int_M u d\mu_h.$$

One can prove that if h_t satisfies the Calabi flow. Then,

$$\frac{d}{dt}\mathrm{Ca}(h_t)\leq 0,$$

$$\frac{d}{dt}\mathrm{Ent}(h_t)\leq 0.$$

In particular, Calabi energy and the entropy are decreasing along the flow. In order to prove the second lemma, we need to assume that there exists a metric of non positive constant Gaussian curvature on M.

Hence, the entropy and Calabi energy are uniformly bounded along the flow

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Therefore, we can apply the following modification of Chen-Cheng theorem to prove that in finite time, the flow is uniformly bounded in $W^{2,2}$.

Theorem 3.1. *Let $h_n = e^{u_n}\tilde{h}$ be a sequence of metrics with volume 1. Assume that there exists a constant A such that*

$$Ca(h_n) \leq A,$$
$$Ent(h_n) \leq A.$$

Then, there exists a constant $C = C(\tilde{h}, A)$ such that

$$||u_n||_{W^{2,2}} \leq C.$$

All this in place, we can sketch the proof as follows.

As mensioned before, the short time existence is standard. Suppose there exists $T < \infty$ is the maximum number such that the flow h_t exists for $t \in [0, T)$.

Since Calabi energy and the entropy are uniformly bounded along the flow, there exists a constant $C = C((\tilde{h}, h_0, T)$ such that

$$||u_t||_{W^{2,2}} \leq C, \quad \text{for any} \quad t \in [0, T).$$

In paericular, there exists $t_n \to T^-$ such that

$$u_{t_n} \to u_\infty \ \text{ in } \ W^{1,2}.$$

By the short time existence, there exists a positive ϵ such that Calabi flow $g_t = e^{v_t}\tilde{h}$ starting with $v_0 = u_\infty$ exists for $t \in [0, \epsilon)$. Therefore, for $t \in [T, T + \epsilon)$, we can define $h_t = g_{T+t}$. It is standard that h_t satisfies the Calabi flow for $t \in [0, T + \epsilon)$ which is a contradiction with the maximality of T.

References

[1] E. Calabi, Extremal Kähler metrics, Seminar on Differential Geometry, *Ann. of Math. Stud.*, No. 102, Princeton Univ. Press, Princeton, N. J., 1982 259–290.

[2] P. T. Chruściel, Semi-global existence and convergence of solutions of the Robinson-Trautman (2-dimensional Calabi) equation, *Commun. Math. Phys.*, **137** no. 2 (1991) 289–313.

[3] X. Chen and J. Cheng, On the constant scalar curvature Kähler metrics (I)—A priori estimates, *J. Amer. Math. Soc.*, **34** no. 4 (2021) 909–936.

[4] X. Chen and J. Cheng, On the constant scalar curvature Kähler metrics (II)—Existence results, *J. Amer. Math. Soc.*, **34** no. 4 (2021) 937–1009.

[5] Z. Lu and R. Seyyedali, Remarks on a result of Chen-Cheng, *Complex Manifolds*, **11** no. 1 (2024) 12 pp.

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