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THE BOCHNER-EBERLEIN-DOSS PROPERTY FOR THE SEMIGROUP ALGEBRA $\ell^1(\mathbb{Z}^2, \max)$

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Abstract. In recent years, and in the continuation of BSE algebras, the concept of BED algebra for a commutative and semisimple Banach algebra $\mathcal{A}$ has been introduced and studied by some authors. It has also been proved that if $\mathcal{A}$ is regular and also contains a bounded approximate identity $\{e_\alpha\}_\alpha$ such that for every α, the Gelfand transformation e_α, i.e. $\widehat{e_\alpha}$ has compact support, then $\mathcal{A}$ is a BSE algebra if and only if it is a BED algebra. In other words, for such algebras two concepts of BSE and BED coincide. In this paper, let us assume that S is the commutative semigroup $(\mathbb{Z}^2, \max)$. In recent years, it has been proven that the semigroup algebra $\ell^1(\mathbb{Z}^2, \max)$ is a BSE algebra. In this paper, we show that $\ell^1(\mathbb{Z}^2, \max)$ contains a bounded approximate identity with the condition mentioned above. We also show that $\ell^1(\mathbb{Z}^2, \max)$ is regular. Thus, as a main result, we obtain that $\ell^1(\mathbb{Z}^2, \max)$ is also a BED algebra.

1. **Introduction**

The history of the introduction and study of BSE algebras dates back to 1930, when Bochner, Schoenberg, and Eberlein were independently and separately investigating analytic functions. This extensive research eventually led to the proof of an important theorem, known as the Bochner-Schoenberg-Eberlein theorem. In this theorem, a necessary and sufficient condition is given for a

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continuous and bounded function defined on the dual of a locally compact Hausdorff abelian group G to be the Fourier-stieltjes transform of a bounded complex measure. It should be noted that the necessary introduction and information about locally compact abelian groups and their duals, as well as measure algebras and the Fourier-stieltjes transform of measures can be found in [12].

Then in 1990, Takahashi and Hattori extended this topic and study the commutative and without order Banach algebras that were valid in generalizations of Bochner-Schoenberg-Eberlein theorem [16]. In fact, the concept of BSE algebras was first introduced by these two mathematicians and in the following of the research work of Bochner, Schoenberg and Eberlein. the recent papers on BSE algebras are [1], [2], and [6].

BED algebras are also an important class of commutative and semisimple Banach algebras and take their name from the first letters of the names of three mathematicians, Bochner, Eberlein and Doss, who studied and researched in this field. First, Bochner proposed and studied the BED property in 1933, using the theory of harmonic functions without considering a name for these algebras. Then, Schoenberg in 1942, using Bochner's achievements, investigated further the properties of these algebras. Finally, in 1967 Doss, using the theories and achievements of Bochner and Schoenberg, proved a fundamental theorem. This theorem, in fact, characterizes the absolute continuous measures on the locally compact Hausdorff abelian group G and will be discussed later. Various forms of this theorem and its results are detailed in [9], [10], and [11].

Function $\tau \in C_b(\widehat{G})$ is the Fourier-stieltjes transform of a absolutely continuous complex measure if and only if the following conditions hold;

(i). there exists $L > 0$ such that for any $q \in \operatorname{span}(\widehat{G})$ as

$$q(x) = \sum_{i=1}^{n} c_i(-x, \gamma_i)$$

and with $\|q\|_\infty \leq 1$ we have

$$\left| \sum_{i=1}^{n} c_i \tau(\gamma_i) \right| \leq L.$$

(ii). For any $\varepsilon > 0$, there exists a compact subset K of $\widehat{G}$ such that for any $q \in \operatorname{span}(\widehat{G} \backslash K)$ as

$$q(x) = \sum_{i=1}^{n} c_i(-x, \gamma_i)$$

and with $\|q\|_\infty \leq 1$ we have

$$\left| \sum_{i=1}^{n} c_i \tau(\gamma_i) \right| < \varepsilon.$$

In 2007, Inoue and Takahashi, using Doss's achievements, which actually characterized the Gelfand image of the group algebra $L^1(G)$ and presented a new method for characterizing the Gelfand image

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of any arbitrary commutative and semisimple Banach algebra. The research of these authors was actually conducted around Doss' results. In this way, BED algebras were introduced [14]. As the recent works on the BED we refer to [3], [4] and [5].

To introduce BED algebras, we need some introductions, which will be given below. The most complete references in this regard can be mentioned in references [14] and [15].

Let $\mathcal{A}$ be a commutative Banach algebra. A bounded net $\{x_\alpha\}_\alpha$ in $\mathcal{A}$ is a bounded approximate identity for $\mathcal{A}$ if $\lim_\alpha \|ax_\alpha - a\| = 0$, for each $a \in \mathcal{A}$. Suppose that $\mathcal{A}^*$ and $\Delta(\mathcal{A})$ are the dual space and the character space of $\mathcal{A}$, respectively. Then $\Delta(\mathcal{A})$ is always a locally compact Hausdorff space, equipped with the weak* topology, inherited from $\mathcal{A}^*$, called Gelfand topology. Suppose that $\operatorname{span}(\Delta(\mathcal{A}))$ is the linear span of $\Delta(\mathcal{A})$ in $\mathcal{A}^*$. Note that each $q \in \operatorname{span}(\Delta(\mathcal{A}))$ has the unique expression as

$$q = \sum_{\varphi\in\Delta(\mathcal{A})} \widehat{q}(\varphi)\varphi,$$

in which $\widehat{q}$ is a unique complex-valued function on $\Delta(\mathcal{A})$, with finite support. Moreover as usual, $C_b(\Delta(\mathcal{A}))$ denotes the Banach algebra of all complex-valued bounded and continuous functions on $\Delta(\mathcal{A})$, under the supremum norm and pointwise product.

We also recall the Gelfand representation of $\mathcal{A}$, which is a norm-decreasing algebra homomorphism, defined as

$$\Gamma_{\mathcal{A}} : \mathcal{A} \to C_b(\Delta(\mathcal{A})) \qquad a \mapsto \widehat{a},$$

where $\widehat{a}(\varphi) = \varphi(a)$ $(\varphi \in \Delta(\mathcal{A}))$. The image of $\Gamma_{\mathcal{A}}$ is denoted by $\widehat{\mathcal{A}}$. We say that $\mathcal{A}$ is semisimple if $\Gamma_{\mathcal{A}}$ is injective. Furthermore, $\mathcal{A}$ is called regular if for any closed subset E of $\Delta(\mathcal{A})$ and each $\varphi_0 \in \Delta(\mathcal{A}) \setminus E$, there exists $a \in \mathcal{A}$ such that $\widehat{a}(\varphi_0) = 1$ and $\widehat{a}(\varphi) = 0$, for all $\varphi \in E$.

A function $g \in C_b(\Delta(\mathcal{A}))$ is called a BSE function if there exists a constant $L > 0$ such that for every finite number of $\varphi_1, \dots, \varphi_n$ in $\Delta(\mathcal{A})$ and the same number of complex numbers $d_1, \dots, d_n$, the inequality

$$\left|\sum_{j=1}^{n} d_j g(\varphi_j)\right| \le L \left\|\sum_{j=1}^{n} d_j \varphi_j\right\|_{\mathcal{A}^*}$$

holds. The BSE norm of g ($\|g\|_{\mathrm{BSE}}$) is the infimum of all such L. The set of all BSE functions is denoted by $C_{\mathrm{BSE}}(\Delta(\mathcal{A}))$. Moreover, $C_{\mathrm{BSE}}(\Delta(\mathcal{A}))$ is a commutative and semisimple Banach algebra under the norm $\|\cdot\|_{\mathrm{BSE}}$ and pointwise product [16, Lemma 1]. Now for the commutative and semisimple Banach algebra $\mathcal{A}$, let

$$\mathcal{M}(\mathcal{A}) = \{\Phi \in C_b(\Delta(\mathcal{A})) :\ \Phi.\widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}\}.$$

The members of $\mathcal{M}(\mathcal{A})$ are called multiplier. Following [16], $\mathcal{A}$ is called a BSE algebra if $\mathcal{M}(\mathcal{A}) = C_{\mathrm{BSE}}(\Delta(\mathcal{A}))$. In the sequel, we will introduce BED algebras. For each $g \in C_{\mathrm{BSE}}(\Delta(\mathcal{A}))$ and any

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compact subset K of $\Delta(\mathcal{A})$, let

$$\|g\|_{\mathrm{BSE,K}} = \sup\left\{\left|\sum_{\varphi\in\Delta(\mathcal{A})} \widehat{q}(\varphi)g(\varphi)\right| :\ q \in \operatorname{span}(\Delta(\mathcal{A})), \|q\|_{\mathcal{A}^*} \leq 1, \widehat{q}(\varphi) = 0\ (\varphi \in K)\right\}.$$

Moreover, suppose that $\|g\|_{\mathrm{BSE},\infty} = \inf_K \|g\|_{\mathrm{BSE,K}}$ and

$$C^0_{\mathrm{BSE}}(\Delta(\mathcal{A})) = \{g \in C_{\mathrm{BSE}}(\Delta(\mathcal{A})) :\ \|g\|_{\mathrm{BSE},\infty} = 0\}.$$

Then $C^0_{\mathrm{BSE}}(\Delta(\mathcal{A}))$ is always a closed ideal of $C_{\mathrm{BSE}}(\Delta(\mathcal{A}))$ [14, Corollary 3.9]. By [?, Definitions 2.1 and 2.2], $C_{\mathrm{BSE}}(\Delta(\mathcal{A}))$ and $C^0_{\mathrm{BSE}}(\Delta(\mathcal{A}))$ are called the BSE−extension and the Inoue-Doss ideal associated with $\mathcal{A}$, respectively. Following [14], a commutative and semisimple Banach algebra $\mathcal{A}$ is called a BED algebra if $\widehat{\mathcal{A}} = C^0_{\mathrm{BSE}}(\Delta(\mathcal{A}))$. Note all BED algebras has the BSE norm property in the sense that there exists $K > 0$ such that $\|a\| \leq K\|\widehat{a}\|_{\mathrm{BSE}}$ $(a \in \mathcal{A})$. We refer the readers to [14], in order to achieve several necessary and sufficient conditions for establishing the equality of $\widehat{\mathcal{A}}$ and $C^0_{\mathrm{BSE}}(\Delta(\mathcal{A}))$. Moreover, we refer to [3], [4] and [5] as the recent works in this context.

In this paper, we study the BED property for the semigroup algebra $\ell^1(\mathbb{Z}^2, \max)$. Our approach for this investigation is to apply the results of [8] together with [14, Theorem 4.7]. Indeed, it has been proved in [14, Theorem 4.7] that if $\mathcal{A}$ is a regular Banach algebra which possesses a bounded approximate identity $\{e_\lambda\}_{\lambda\in I}$ such that each $\widehat{e_\lambda}$ $(\lambda \in I)$ has compact support, then $\mathcal{A}$ is a BSE algebra if and only if $\mathcal{A}$ is of BED. Furthermore, $\ell^1(\mathbb{Z}^2, \max)$ is a BSE algebra by [8]. Here, we verify the conditions of [14, Theorem 4.7] for $\ell^1(\mathbb{Z}^2, \max)$ to prove by using [8] that $\ell^1(\mathbb{Z}^2, \max)$ is a BED algebra, as well.

2. Main Results

In this section, we provide our main result. Before, we precisely introduce the semigroup algebra $\ell^1(\mathbb{Z}^2, \max)$; see [8] and [13], for more information about semigroup algebras.

The semigroup $(\mathbb{Z}^2, \max)$ is the set $\mathbb{Z}^2$ with coordinatwise maximum operation, defined as

$$(m_1, n_1)(m_2, n_2) = (\max\{m_1, m_2\}, \max\{n_1, n_2\}),$$

for all $(m_1, n_1), (m_2, n_2) \in \mathbb{Z}^2$. It is clear that $(\mathbb{Z}^2, \max)$ is abelian. The ℓ^1−algebra $\ell^1(\mathbb{Z}^2, \max)$ is a linear space, defined as

$$\ell^1(\mathbb{Z}^2, \max) = \left\{f : \mathbb{Z}^2 \to \mathbb{C};\ \|f\|_1 = \sum_{(m,n)\in\mathbb{Z}^2} |f(m, n)| < \infty\right\},$$

which is a Banach algebra under the norm $\|\cdot\|_1$ and the convolution product, defined as

$$f * g(m, n) = \sum_{(m_1,n_1)(m_2,n_2)=(m,n)} |f(m_1, n_1)g(m_2, n_2)| \qquad ((m, n) \in \mathbb{Z}^2),$$

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for all $f, g \in \ell^1(\mathbb{Z}^2, \max)$. The character space of $\ell^1(\mathbb{Z}^2, \max)$ is introduced, as follows.

Let $\mathbb{Z}_\infty = \mathbb{Z} \cup \{\infty\}$ and for each $n \in \mathbb{Z}$, $n < \infty$. For every $n \in \mathbb{Z}$, let

$$(n, \infty] = \{m \in \mathbb{Z} : \ n < m\} \cup \{\infty\}.$$

Moreover, suppose that

$$\mathcal{B} = \{\{m\} : \ m \in \mathbb{Z}\} \cup \{(n, \infty] : \ n \in \mathbb{Z}\},$$

which is clearly a basis for a topology on $\mathbb{Z}_\infty$. Then $\mathbb{Z}_\infty$ is equipped with this topology and we consider $\mathbb{Z}_\infty^2 = \mathbb{Z}_\infty \times \mathbb{Z}_\infty$, with the product topology. It should be noted that any open neighborhood of ∞, contains an open set $(n, \infty]$ $(n \in \mathbb{Z})$. Moreover, it is easily verified that for each $n \in \mathbb{Z}$, the set $[n, \infty]$ is compact in $\mathbb{Z}_\infty$.

By [8, Lemma 2.8], $\Delta(\ell^1(\mathbb{Z}^2, \max))$ is homeomorphic with $\mathbb{Z}_\infty^2$. Indeed,

$$\Delta(\ell^1(\mathbb{Z}^2, \max)) = \left\{\varphi_{(M,N)} : \ \ (M, N) \in \mathbb{Z}_\infty^2\right\},$$

where for each $(M, N) \in \mathbb{Z}_\infty^2$ we have

$$\varphi_{(M,N)}(f) = \sum_{n=-\infty}^{M} \sum_{m=-\infty}^{N} f(m, n) \qquad (f \in \ell^1(\mathbb{Z}^2, \max)).$$

Note that Gelfand topology on $\Delta(\ell^1(\mathbb{Z}^2, \max))$ is coincided with the product topology on $\mathbb{Z}_\infty^2$, discussed above; see [8, Lemma 2.8] for more details.

Following [13], an abelian semigroup S is called separating if for all $s, t \in S$, the equations $s^2 = t^2 = st$ imply $s = t$. By [13, Theorem 5.8], the semigroup algebra $\ell^1(S)$ is semisimple if and only if S is separating. As stated in [8], it is easily verified that $(\mathbb{Z}^2, \max)$ is separating. It follows that $\ell^1(\mathbb{Z}^2, \max)$ is semisimple.

Recall that for each $(m, n) \in \mathbb{Z}^2$, the complex-valued function $\delta_{(m,n)}$ is defined on $\mathbb{Z}^2$ as

$$\delta_{(m,n)}(l, k) = \begin{cases} 1 & (l, k) = (m, n) \\ 0 & (l, k) \neq (m, n). \end{cases}$$

Let S be the semigroup $(\mathbb{Z}^2, \max)$. Then for all $m, n \in \mathbb{Z}$, supp $\widehat{\delta_{(m,n)}} \subseteq [m, \infty] \times [n, \infty]$. In particular, $\widehat{\delta_{(m,n)}}$ has compact support.

Let S be the semigroup $(\mathbb{Z}^2, \max)$. Then for each $(M_0, N_0) \in \mathbb{Z}^2$ there exists $f \in \ell^1(S)$ such that $\widehat{f}(\varphi_{(M_0,N_0)}) = 1$ and $\widehat{f}(\varphi_{(M,N)}) = 0$, for all $(M, N) \in \mathbb{Z}_\infty^2$ with $(M, N) \neq (M_0, N_0)$.

Let S be the semigroup $(\mathbb{Z}^2, \max)$. Then for each closed subset E of $\mathbb{Z}_\infty^2$ with $(\infty, \infty) \notin E$, there exists $f \in \ell^1(S)$ such that $\widehat{f}(\varphi_{(\infty,\infty)}) = 1$ and $\widehat{f}(\varphi_{(M,N)}) = 0$, for all $(M, N) \in E$.

Let S be the semigroup $(\mathbb{Z}^2, \max)$. Then for each closed subset E of $\mathbb{Z}_\infty^2$ and $M_0 \in \mathbb{Z}$ with $(M_0, \infty) \notin E$, there exists $f \in \ell^1(S)$ such that $\widehat{f}(\varphi_{(M_0,\infty)}) = 1$ and $\widehat{f}(\varphi_{(M,N)}) = 0$, for all $(M, N) \in E$.

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In fact since E is closed and $(M_0,\infty)\notin E$, there exists $N_0\in\mathbb{Z}$ such that

$$\{M_0\}\times(N_0,\infty]\cap E=\varnothing. \tag{2.1}$$

Define the complex-valued function f on S as

$$f(m,n)=\begin{cases}1 & (m,n)=(M_0,N_0+1)\\ -1 & (m,n)=(M_0+1,N_0+1)\\ 0 & \text{otherwise,}\end{cases}$$

which obviously belongs to $\ell^1(S)$. Moreover,

$$\widehat{f}(\varphi_{(M_0,\infty)})=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{M_0}f(m,n)=f(M_0,N_0+1)=1.$$

Also regarding to equation (2.1), for all $(M,N)\in E$ four cases occur. If $M=M_0$ and $N\leq N_0$, then it is easily observed that $\widehat{f}(\varphi_{(M_0,N)})=0$. Similarly, if $M<M_0$ and $N\in\mathbb{Z}_\infty$, then $\widehat{f}(\varphi_{(M,N)})=0$. Moreover, if $M\geq M_0+1$ and $N\geq N_0+1$, then

$$\widehat{f}(\varphi_{(M,N)})=f(M_0+1,N_0+1)+f(M_0,N_0+1)=-1+1=0.$$

Finally, for the case $M\geq M_0+1$ and $N\leq N_0$, it is obviously obtained that $\widehat{f}(\varphi_{(M,N)})=0$.

Let S be the semigroup $(\mathbb{Z}^2,\max)$. Then for each closed subset E of $\mathbb{Z}^2_\infty$ and $N_0\in\mathbb{Z}$ with $(\infty,N_0)\notin E$, there exists $f\in\ell^1(S)$ such that $\widehat{f}(\varphi_{(\infty,N_0)})=1$ and $\widehat{f}(\varphi_{(M,N)})=0$, for all $(M,N)\in E$.

The semigroup algebra $\ell^1(\mathbb{Z}^2,\max)$ is regular.

The semigroup algebra $\ell^1(\mathbb{Z}^2,\max)$ is a BED algebra.

In fact by [8], $\{\delta_{(-n,-n)}\}_{n\in\mathbb{N}}$ is a bounded approximate identity for $\ell^1(\mathbb{Z}^2,\max)$. Moreover, $\widehat{\delta_{(-n,-n)}}$ $(n\in\mathbb{N})$ has compact support. In addition, $\ell^1(\mathbb{Z}^2,\max)$ is regular. Since $\ell^1(\mathbb{Z}^2,\max)$ is a BSE algebra [8], it follows $\ell^1(\mathbb{Z}^2,\max)$ is a BED algebra, as well.

The semigroup algebra $\ell^1(\mathbb{Z}^2,\max)$ has BSE norm property.

3. Conclusions

Let S be the abelian semigroup $(\mathbb{Z}^2,\max)$. Recently, it was proved that the semigroup algebra $\ell^1(\mathbb{Z}^2,\max)$ is a BSE algebra. Here, we establish that $\ell^1(\mathbb{Z}^2,\max)$ is a BED algebra, as well.

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