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HISTORICAL APPROACHES AND MODERN METHODS IN ANALYZING THE BRACHISTOCHRONE PROBLEM

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Abstract. This paper presents the history of competition of mathematicians to uncover the characteristics of the cycloid curve, titled "Helen of Geometry". It also includes historical notes related to the Brachistochrone problem. The cycloid is introduced in parametric form, and its properties and applications are obtained. Additionally, the calculus of variations and the study of the Brachistochrone problem, along with various solutions, are presented. We show that the optimal solution to the Brachistochrone problem is the cycloid curve. Several procedures in Maple software are introduced to provide a more accurate analysis. These graphical outputs help in understanding the yields. The proposed Maple procedures allow for the comparison of the shortest time problem along paths such as lines, circular arcs, and cycloids. These procedures also aid in drawing connecting curves between points and in calculating optimal travel times numerically. The findings clearly demonstrate that the travel time along the cycloid curve is significantly shorter than along other paths.

1. Introduction and Historical Background

Undoubtedly, the significant achievements of humanity across various areas of science and technology rely on the focus given to fundamental sciences, particularly mathematics. UNESCO's decision to acknowledge the year 2000 as the 'International Year of Mathematics'[1] and the year 2022 as

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[1]world mathematical year 2000

the *"International Year of Basic Sciences for Sustainable Development"*[2] illustrates this. The goals that have motivated these declarations have been to enhance awareness of the fundamental sciences as the foundation for the progress and growth of communities.

The study of leading nations in high technologies reveals a significant emphasis on advancing fundamental sciences, particularly mathematics. Mathematics occupies a crucial role in fostering creativity, logical reasoning, and systematic thinking, with its wide-ranging applications across engineering, computer science, aerospace, military science, and economics. Galileo Galilei[3] referred to mathematics as the "language of nature", while Gauss[4] labeled it the "queen of sciences" [1]. This field not only endures but also stands as one of humanity's greatest achievements.

The mathematical sciences originated from humanity's needs and observations of nature and gradually expanded with the advancement of societies. Mathematicians first sought to uncover the laws of nature and later formulated precise models, paving the way for scientific and technological progress [12]. One of the prominent subjects in the history of mathematics is the study of the *cycloid*[5], a curve celebrated for its geometric beauty and unique properties. Due to its significance, it is famously known as the *"Helen of Geometry"*[6] [1, 8].

The cycloid is the path followed by a point on the edge of a circle as it rolls along a line path. This curve captivates mathematicians and has also been a platform for scientific competition among leading academics. Historical research suggests that Galileo was the pioneer in showing, through a clever geometric approach, that the area beneath one arch of the cycloid is three times larger than that of its generating circle [11].

After Galileo, his student Vincenzo Viviani[7] and Marin Mersenne[8] developed an interest in exploring this curve. Mersenne's computations showed that the horizontal distance between two points on the cycloid is equal to the circumference of the circle that generates it. He also proposed that the curve might conform to a semi-ellipse [8].

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[2]international year of basic sciences for sustainable development

[3]Galileo Galilei (1564-1642)

[4]Johann Carl Friedrich Gauss (1777-1855)

[5]Cycloid

[6]Helen of Geometry

[7]Vincenzo Viviani (1622-1703)

[8]Marin Mersenne (1588-1648)

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Additional inquiries and discussions were conducted by mathematicians like Roberval[9], Descartes[10], and Fermat[11]. Roberval, applying *Cavalieri's principle*[12], determined the area beneath a single arch of the cycloid and demonstrated that it is three times the area of the circle that generates it. In addition, he created a technique for finding tangents to the curve and determining the volume of the solid generated by rotating one arch [11].

In the latter half of the 17th century, Blaise Pascal[13] conducted a study of the geometric properties of cycloid, driven by personal motivation. Pascal presented several new problems related to this curve in the form of a scientific competition. Prominent mathematicians, including John Wallis[14], participated in the challenge; however, none were able to provide a complete solution. These competitions significantly contributed to the advancement of geometric methods and later facilitated the development of calculus [1].

The endeavors to uncover the properties of the cycloid laid the groundwork for the emergence of the *calculus of variations*, a branch of mathematics that investigates the optimization of functionals. This field has played a pivotal role in solving dynamic optimization problems and optimal control theory. Consequently, the study of the cycloid is significant not only from a historical perspective but also in terms of its contributions to mathematical methods and modern applications [6, 7].

2. **The Cycloid Curve**

The cycloid is he famous curve in mathematics, with its roots tracing back to the early 16th century. Defined as the trajectory of a point on the circumference of a circle rolling along a line without slipping, the cycloid has captured both mathematical intrigue and practical applications. Galileo named this curve the *cycloid* and proposed its use in the design of bridge arches. This concept became feasible with the advent of reinforced concrete, establishing the cycloid as a reliable choice in structural engineering [3, 9].

The cycloid curve can be mathematically described using the following parametric form [10]:

$$\left\{\begin{array}{l} x(t)=a(t-\sin t), \\ y(t)=a(1-\cos t), \quad 0 \leq t \leq 2\pi, \end{array}\right. \tag{2.1}$$

where a is the radius of the generating circle. Depending on the position of the tracing point relative to the center of circle, this curve can take one of three forms:

[9]Gilles Personne de Roberval (1602-1675)

[10]René Descartes (1596-1650)

[11]Pierre de Fermat (1607-1665)

[12]Cavalieri's principle

[13]Blaise Pascal (1623-1662)

[14]John Wallis (1616-1703)

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- **Ordinary Cycloid**: The point lies on the circumference of the circle.
- **Curtate Cycloid**: The point lies inside the circle.
- **Prolate Cycloid**: The point lies outside the circle.

2.1. **Geometric and Physical Properties.** The cycloid possesses numerous fascinating properties that are highly relevant to both mathematics and physics:

(1) **Arc Length**: The length of one complete arch is $8a$, which is eight times the radius of circle.
(2) **Area Under the Curve**: The area enclosed by one arch and the baseline is $3\pi a^2$, equivalent to three times the area of the generating circle.
(3) **Dimensions**: The height of the cycloid is equal to the diameter of circle ($2a$), and its base is the circumference of circle ($2\pi a$).
(4) **Radius of Curvature**: At any given point, the radius of curvature is twice the distance between the point and the center of rotation.

2.2. **Applications of the Cycloid Curve.** The cycloid has applications in a variety of scientific and engineering based on its unique geometric and physical characteristics:

- **Structural Design**: The cycloid serves as an ideal model for designing arches in bridges and other robust structures.
- **Optimal Paths**: The cycloid is the solution to the Brachistochrone problem, representing the fastest descent between two points under gravity.
- **Pendulum Clocks**: Cycloidal arcs ensure consistent oscillation periods, independent of amplitude.
- **Physics of Particles**: The trajectory of charged particles in orthogonal electric and magnetic fields forms a cycloid.

3. The Brachistochrone Problem

The brachistochrone problem, famous for its intrinsic appeal and its pivotal role in shaping the calculus of variations, holds a significant place in the annals of mathematical history. This problem posits a frictionless bead sliding from a higher point A to a lower point B. The query is: which path, a line or an arbitrary curve connecting the two points, would minimize the time taken for the bead to reach B? [2] Intuitively, the line, being the shortest distance between two points, might seem the quickest. However, a deeper analysis reveals that this assumption might be incorrect. In fact, the bead could potentially cover a longer path in less time if it were to initially fall vertically and then continue its motion [10]. This concept was historically proposed by Galileo. In 1694, Johann Bernoulli posed a more commone version of this problem, now known as the brachistochrone problem.

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To model this problem, we consider the motion of a bead under gravity along an arbitrary curve connecting two points A and B. The primary objective is to determine the equation of the curve that minimizes the time taken for the bead to travel from A to B. By applying the principles of conservation of energy and the velocity of the bead along the curve, an equation for the time taken along the path is derived [3, 9]:

$$T[y] = \frac{1}{\sqrt{2g}} \int_0^a \frac{\sqrt{1+(y'(x))^2}}{\sqrt{y}} dx, \tag{3.1}$$

where g is the acceleration due to gravity . By the Euler-Lagrange equation, the differential equation corresponding to the desired curve is obtained as follows [5]:

$$y\left(1+\left(y'\right)^2\right) = c, \tag{3.2}$$

where c is a constant.

Furthermore, the cycloid is introduced as a particular curve to solve the brachistochrone problem. The distinctive feature of this curve, specifically recognized as the curve of fastest descent, is that the motion of the bead along this curve from point A to point B always occurs in a fixed and minimal time. This curve also finds practical applications in the design of pendulum clocks and timekeeping instruments. The differential equation associated with the cycloid is solved, demonstrating that it is indeed the unique curve that guarantees the shortest time for motion between two points. The solutions to the differential equations yield the parametric equations of the cycloid. This curve, recognized as the curve of shortest time between two points, has found extensive applications in various scientific and engineering analyses [4].

4. Analysis of the Shortest Time Problem Along Different Paths

This section examines the **shortest time problem** for three distinct paths (cycloid, line, and circular arc) demonstrating that the cycloid minimizes time. The parametric equations of cycloid and their role in solving the Brachistochrone Problem highlight its optimality. Using the Maple procedure, the analysis compares travel times numerically and visually across paths, confirming the efficiency of cycloid and reinforcing its importance in both theoretical and practical applications.

4.1. **Cycloid Path Analysis.** The cycloid, known as the solution to the Brachistochrone Problem, is defined by the parametric equations (2.1) where $a = r$ is the radius of the generating circle and $0 \leq t \leq t_1$. The solution is obtained numerically for given endpoints $B(x_1, y_1)$. The results for several cases are shown in Table 1.

Table 1. Time Calculations for Cycloid Paths

Example	Endpoint (x_1, y_1)	Time to B	Time to Minimum Point
I	$(1, 1)$	0.5829950146	0.7593384715
II	$(3, 1)$	1.018486861	0.7897247281
III	$(1, 2)$	0.6940765647	1.555971031
IV	$(3, 2)$	0.9806021830	1.003869908

Table 2. Time Calculations for Line Paths

Example	Endpoint (x_1, y_1)	Time to B
I	$(1, 1)$	0.6386599136
II	$(3, 1)$	2.473519209
III	$(1, 2)$	0.7140434906
IV	$(3, 2)$	1.994214941

Table 3. Time Calculations for Circular Paths

Example	Endpoint (x_1, y_1)	Time to B
I	$(1, 1)$	0.5920615866
II	$(3, 1)$	1.057726406
III	$(1, 2)$	0.6978216999
IV	$(3, 2)$	1.004460007

4.2. **Line Path Analysis.** The equation for a line passing through $A(0, 0)$ and $B(x_1, y_1)$ is:

$$y = mx, \quad m = \frac{y_1}{x_1}. \tag{4.1}$$

The travel time along this path is:

$$T = \sqrt{\frac{2(x_1^2 + y_1^2)x_1}{y_1 g}}, \tag{4.2}$$

where g is the gravitational acceleration.

4.3. **Circular Path Analysis.** For a circular trajectory centered at $(r, 0)$ with radius r, passing through $B(x_1, y_1)$, the circle is defined as:

$$(x - r)^2 + y^2 = r^2, \quad r = \frac{x_1^2 + y_1^2}{2x_1}. \tag{4.3}$$

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TABLE 4. Comparison of Travel Times

Path Type	Time to $(2,3)$	Time to $(5,3)$
Line	1.329476627	3.399526989
Circle	0.9045011664	1.299965875
Cycloid	0.8967041001	1.266150837

4.4. **Comparison of Paths.** The results of time comparisons for cycloid, line, and circular paths are summarized in Table 4. The cycloid consistently provides the shortest travel time.

Also, the various paths traveled between the initial and final points are plotted in comparison to the optimal path of the cycloid curve in Figure (1). This comparison enables the evaluation of the results along the line and circular paths relative to the optimal trajectory.

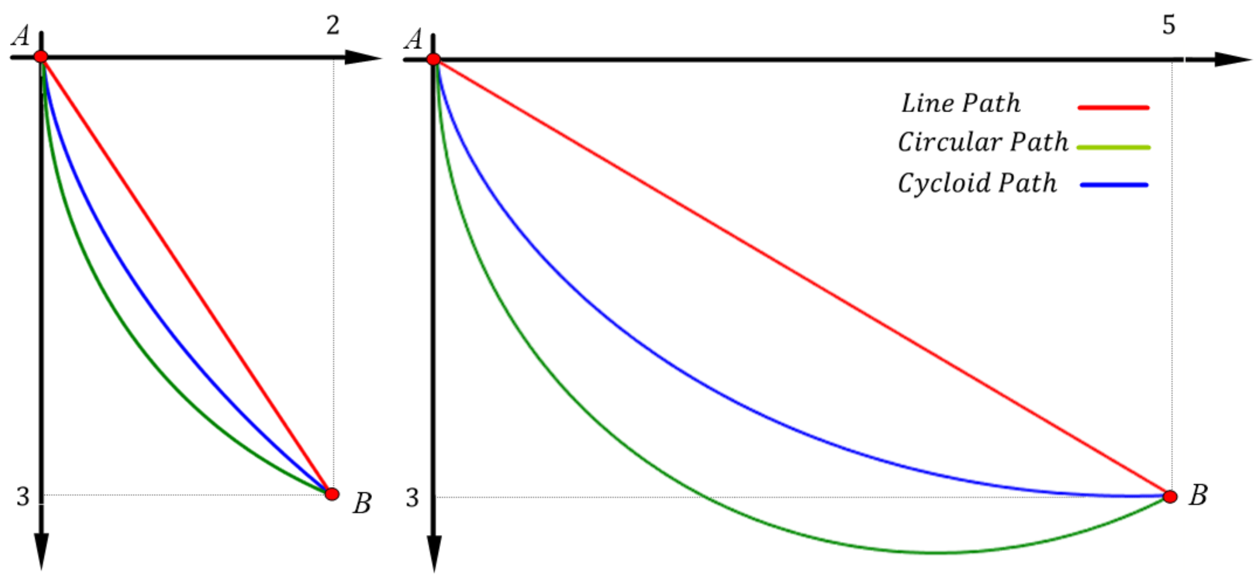

FIGURE 1. Comparison of paths between $(0,0)$ to $(2,3)$ and $(0,0)$ to $(5,3)$

5. CONCLUSION

This paper explores the historical and mathematical significance of the Brachistochrone problem, highlighting the cycloid as the optimal solution. It discusses the development of variational calculus, the various methods used to solve the shortest time problem, and introduces Maple procedures for analyzing optimal paths and comparing results numerically and graphically.

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