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A SURVEY ON BASIC POLYGROUPS AND THEIR AUTOMORPHISMS

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Abstract. Polygroups have been applied in many areas, such as geometry, lattices, combinatorics and color scheme. In this paper, we study a special type of polygroups; basic polygroups and present some of their properties. First, we determine the finite basic polygroup. We define the concept of characteristic by using the automorphism of a polygroup and prove that the basic set is a characteristic subpolygroup. A connection between basic polygroups associated with basic set is also investigated. The article provides valuable examples and results for researchers who study polygroups.

1. Introduction

The hyperstructure theory was firstly introduced by F. Marty at the 8th congress of Scandinavian Mathematicians in 1934. Marty introduced the concept of hypergroups as a generalization of groups and used it in different contexts like algebraic functions, rational fractions and non-commutative groups. In calssical algebraic structures, the synthetic result of two elements is an element, while in the hyper algebraic system, the synthetic result of two elements is a set of elements, therefore it can be said that the notion of hyperstructures is a generalization of classical algebraic structures, from this point of view. Hyperstructures have many applications to several sectors of both pure and applied sciences as geometry, graphs and hypergraphs, fuzzy sets and rough sets, automata, cryptography,

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codes, relation algebras, C-algebras, artificial intelligence, probabilities, chemistry, physics, especially in atomic physics and in harmonic analysis.

After the emergence of algebraic hyperstructures, many scientists around the world studied and researched in this field. In Iran, scientists such as Borzoei, Zahedi, Davvaz, Ameri, etc , have studied and investigated algebraic and logical hyperstructures and have written valuable articles, books and treatises about them. Some researchers have also written many articles about their application in physics, chemistry and other sciences.

Th polygroups theory is a natural generalization of the group theory. Polygroups have been applied in many areas, such as geometry, lattice theory, combinatorics and color schemes. There exists a rich bibliography: publications appeared within 2012 can be found in [1]. This book contains the principal definitions endowed with examples and the basic results of the theory. Applications of hypergroups appear in special subclasses like polygroups that they were studied by Comer [1], also see [3, 4].

In this paper, we introduce the basic polygroup using the concept of polygroups and study relationship between polygroup and basic polygroups.

2. **Main Results**

In this section, we recall some definitions which we will need in the next sections.

Let H be a non-empty set and $P^*(H)$ be the set of all non-empty subsets of H. Let $\cdot$ be a hyperoperation on H, that is, $\cdot$ is a map from $H \times H$ into $P^*(H)$, and the structure $(H, \cdot)$ is called a hypergroupoid. For any two non-empty subsets A and B of H and $x \in H$, we define $A \cdot B = \bigcup\limits_{\substack{a\in A \\ b\in B}} a \cdot b$, $A \cdot x = A \cdot \{x\}$, $x \cdot B = \{x\} \cdot B$, a hypergroupoid $(H, \cdot)$ is called a hypergroup if $\forall\ a, b, c \in H$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ and $a \cdot H = H \cdot a = H$. The map $f : H \to K$ is called a homomorphism of hypergroups if for all $a, b \in H$, we have $f(a \cdot b) = f(a) * f(b)$.

A homomorphism f is called an isomorphism if f is a one to one and onto map. If H is a hypergroup, an automorphism of H is an isomorphism from H to H. The set of automorphisms of H denoted by $\mathrm{Aut}(H)$.

A hypergroupoid $(P, \cdot)$ is called a polygroup, provided that (i) P be associative, (ii) it has a scaler identity e, (iii) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$, where $^{-1}$ is an unitary operation on H.

A non-empty subset K of P is said to be a subpolygroup of P, if for any $x, y \in K$, $x \cdot y^{-1} \subseteq K$, and is denoted by $K \leq P$. A subpolygroup K of P is said to be characteristic in P if $\alpha(K) \subseteq K$, for all $\alpha \in \mathrm{Aut}(P)$, and we denote it by $K \leq_c P$. Notice, that if K is characteristic in P and $\alpha \in \mathrm{Aut}(P)$, then $\alpha(K) = K$.

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Suppost that P_1 and P_2 are two polygroups and $P_1 \cap P_2 = \{e\}$. Then $(P_1[P_2], *)$ is a polygroup (see [2]) as follows: $x * e = e * x = \{x\}$, for all $x \in P_1 \cup P_2$, and for all $x, y \in P_1 \cup P_2 - \{e\}$,

$$x * y = \begin{cases} xy & if\ x, y \in P_1 \\ x & if\ x \in P_2, y \in P_1 \\ y & if\ x \in P_1, y \in P_2 \\ xy & if\ x, y \in P_2, y \neq x^{-1} \\ xy \cup P_1 & if\ x^{-1} = y \in P_2 \end{cases}$$

Let $\mathcal{U}$ denote the set of all finite product of elements of P. Then $\omega_P = \{x \in P \mid \exists u \in \mathcal{U}\ s.t\ e, x \in u\}$ is called heart of P. In [2] it was rewrote $\beta = \{(x, y) \mid \exists u \in \mathcal{U}\ s.t\ x, y \in u\}$, in addition it was proved that P/β is a group. By using a certain type of equivalence relations, we can connect hypergroups to groups. These equivalence relations are called strong regular relations. More exactly, by a given hypergroup and by using a strong regular relation on it, we can construct a group structure on the quotient set, regular relations provide us new hypergroup structures on the quotient sets.

A Mosayebi Dorcheh introduced R_K on H and it was proved that H/R_K is a k-nilpotent group [5]. Also in [6], Mosayebi, introduced the notion of autonilpotent and autosolvable and investigated some properties of them.

3. Summary of Proofs/Conclusions

In this part of the article, we introduce the basic polygroups and at the end we classify the basic polygroups of order less than 7.

Lemma 3.1. [6] *If P is a polygroup and $\alpha \in Aut(P)$. Then*

(i) $\alpha(e) = e$,
(ii) $\alpha(x^{-1}) = \alpha(x)^{-1}$.

Lemma 3.2. [6] *Let P be a polygroup and let for any $x \in P$, $xx^{-1} = \{e\}$, then P is a group.*

Definition 3.3. *Let (P, e) be a polygroup and $A \subseteq P$, then P is called basic polygroup, if for every two elements $x, y \in P$, such that $|xy| \neq 1$, we have $xy = A$. The set A is called basic set of P. The set of basic polygroups denoted by $P_{m,n}$. Where $m = |A|$ and $n = |P| - |A|$. It is obvious that $m \geq 2$.*

Theorem 3.4. *Let (P, e) be a basic polygroup with basic set A. Then*

(i) $\exists\ x \in P\ s.t\ xx^{-1} = A$,
(ii) *If $a \in A$, then $a^{-1} \in A$,*
(iii) *If $a \in A$ and $aa^{-1} = A$ then $bb^{-1} = e$, for all $b \in A \setminus \{a, a^{-1}\}$,*
(iv) $A = \omega_P$,

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(v) $\forall\ p \in P \setminus A,\ pp^{-1} = A$.

Proof. (i) By Lemma (3.2), there exists $x \in P$ such that $|xx^{-1}| \neq 1$ and by definition, $e \in xx^{-1} = A$.

(ii) Let $xx^{-1} = A$ for some $x \in P$, then $a \in xx^{-1}$ implies $a^{-1} \in xx^{-1} = A$.

(iii) Let $b \in A \setminus \{a, a^{-1}\}$ such that $aa^{-1} = bb^{-1} = A$. Then $a, b \in ab$ so, we have $ab = A$, which is a contradiction.

(iv) By (i) and definition, the proof is obtained.

(v) Suppose, for a contradiction, that there is element $x \in P \setminus A$ such that $xx^{-1} = e$. Then we have $Px \neq P$, a contradiction.

□

4. Classification of basic polygroups

In this section, we classify finite basic polygroups in the general state. We also examine their automorphisms. We start with some preliminary theorems, but before that it should be noted that in the definition of $P_1[P_2]$, if $P_2 = \{e\}$, we contract that $P_1[P_2] = P_1$. In this section, we denote a group of order n with G_n.

Lemma 4.1. *Let $P \in P_{m,n}$ and let A be the basic set of P and $a_1, a_2 \in A$, then $a_1a_2 \subseteq A$.*

Proof. Suppose, for a contradiction, that $a_1a_2 = x$ and $x \in P \setminus A$. This redues to two cases:

(i) $\exists a \in A\ s.t\ aa^{-1} = A$. Thus $ab = ba = a$, for all $b \in A \setminus \{a, a^{-1}\}$ and so $xa = (a_1a_2)a = a$, a contradiction.

(ii) $\forall b \in A,\ bb^{-1} = e \Rightarrow \exists y \in P \setminus A\ s.t\ yy^{-1} = A \Rightarrow \forall a \in A, ay = y \Rightarrow xy = (a_1a_2)y = y \Rightarrow x \in yy^{-1} = A$, a contradiction. Thus we have $A \leqslant P$.

□

Lemma 4.2. *Let $P \in P_{m,n}$ and let A be the basic set of P and $B = \{x \in P \mid xx^{-1} = A\}$, then $\alpha(A) = A$, $\alpha(B) = B$, for all $\alpha \in \mathrm{Aut}(P)$.*

Proof. Let $A = \{e, a_1, \dots, a_m\}$. Then $\alpha(A) = \{e, \alpha(a_1), \dots, \alpha(a_m)\} = A$. Now let $x \in B$. Then $xx^{-1} = A$. Therefore $A = \alpha(A) = \alpha(xx^{-1}) = \alpha(x)\alpha(x)^{-1} \Rightarrow \alpha(x) \in B$. □

Theorem 4.3. *Let $P \in P_{m,0}$, then either $P \cong G_{m-1}[P_{II}]$ or $P \cong G_{m-2}[P_{III}]$.*

Proof. Let $P = A = \{e, a_1, a_2, \dots, a_{m-1}\}$. We consider two cases:

case 1. $a_1a_1 = A$, in this case the table P is as follows:

Now, if we delete the row and column related to a_1, the rest of the table is a group of order $m-1$. Thus with the definition of $G_{m-1}[P_{II}]$, the result is obtained.

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TABLE 1.

$\cdot$	e	a_1	a_2	a_3	$\cdots$	a_{m-1}
e	e	a_1	a_2	a_3	$\cdots$	a_{m-1}
a_1	a_1	A	a_1	a_1	$\cdots$	a_1
a_2	a_2	a_1			$\cdots$	
a_3	a_3	a_1			$\cdots$	
$\vdots$	$\vdots$	$\vdots$			$\cdots$	$\vdots$
a_{m-1}	a_{m-1}	a_1			$\cdots$	

case 2. $a_1a_2 = A$. Since $a_1a_i = a_ia_1 = a_1$, $i = 1, 3, 4, \dots, m-1$ and $a_2a_i = a_ia_2 = a_2$, $i = 2, 3, 4, \dots, m-1$, so if we delete rows and column related to a_1 and a_2, the rest of the table is a group of ordet $m-2$. Now, if $P_{III} = \{e, a_1, a_2\}$, $P \cong G_{m-2}[P_{III}]$.

□

Corollary 4.4. *Let $P \in P_{m,0}$.*

(i) *If $|P \cap B| = 1$, then $P \cong G_{m-1}[P_{II}]$,*

(ii) *If $|P \cap B| \neq 1$, then $P \cong G_{m-2}[P_{III}]$.*

where $B = \{x \in P \mid xx^{-1} = P\}$.

Theorem 4.5. *Let $P \in P_{m,1}$, then either $P \cong G_m[\mathbb{Z}_2]$ or $P \cong P'[\mathbb{Z}_2]$. Where $P' \in P_{m,0}$, also we have* Aut $(P) \cong$ Aut A.

Proof. Suppose that $x \in P \setminus A$. Then $xa = ax = x$, for all $a \in A$. Now, if there exists $a \in A$ such that $aa^{-1} = A = xx$, then there exists $P' \in P_{m,0}$ and $P \cong P'[\mathbb{Z}_2]$, where $\mathbb{Z}_2 = \{e, x\}$. On the other hand, if $aa^{-1} = e$, for every $a \in A$, we have $P \cong G_m[\mathbb{Z}_2]$. □

Corollary 4.6. *Let $P \in P_{m,1}$.*

(i) *If $|A \cap B| = 0$, then $P \cong G_m[\mathbb{Z}_2]$,*

(ii) *If $|A \cap B| = 1$, then $P \cong (G_{m-1}[P_{II}])[\mathbb{Z}_2]$.*

(iii) *If $|A \cap B| \neq 0, 1$, then $P \cong (G_{m-2}[P_{III}])[\mathbb{Z}_2]$.*

where $B = \{x \in P \mid xx^{-1} = A\}$.

Theorem 4.7. *Let $P \in P_{m,2}$, then either $P \cong G_m[\mathbb{Z}_3]$ or $P \cong P'[\mathbb{Z}_3]$. Where $P' \in P_{m,0}$.*

Proof. Suppose that $P \setminus A = \{x, y\}$, then $xy = yx = A$ (other modes are not possible). Thus $xx = ay = ya = y$ and $yy = ax = xa = y$, for all $a \in A$. But the table P becomes the following table by removing the elements of the set $A \setminus \{e\}$.

Therefore $\{e, x, y\} \cong \mathbb{Z}_3$. Now, by the definition of $A[\mathbb{Z}_3]$, the proof is complete. □

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TABLE 2.

$\cdot$	e	x	y
e	e	x	y
x	x	y	e
y	y	e	x

Corollary 4.8. *Let $P \in P_{m,2}$. Then we have*

$$P \cong \begin{cases} G_m[\mathbb{Z}_3] & if|A \cap B| = 0 \\ (G_{m-1}[P_{II}])[\mathbb{Z}_3] & if|A \cap B| = 1 \\ (G_{m-2}[P_{III}])[\mathbb{Z}_3] & if|A \cap B| \neq 0, 1 \end{cases}$$

Corollary 4.9. *Let $P \in P_{m,n}$.*

(i) *If $|A \cap B| = 0$, then $P \cong G_m[P/\beta]$,*
(ii) *If $|A \cap B| = 1$, then $P \cong (G_{m-1}[P_{II}])[P/\beta]$,*
(iii) *If $|A \cap B| \neq 0, 1$, then $P \cong (G_{m-2}[P_{III}])[P/\beta]$.*

Proof. (i) We have $aa^{-1} = e$, $ab = ba = b$, for all $a \in A$ and $b \in P \setminus A$. But the set A is a group of order m. Therefore by definition $G_m[P/\beta]$, we get our claim.

(ii) If $A \cap B = \{a\}$, then $aa = A$ and $bb^{-1} = e$, for all $b \in A$. Thus $A \cong G_{m-1}[P_{II}]$ and so $P \cong (G_{m-1}[P_{II}])[P/\beta]$.

(iii) If $|A \cap B| \neq 0, 1$, then there exists $x \in A \cap B$ such that $x \neq x^{-1}$, $xx^{-1} = A$. Now, by definition P_{III} the proof is complete.

□

5. CONCLUSION

In this article, the concept of expression basic polygroup and some practical examples of it were given. The following items have been obtained from them:

(i) Basic polygroups of order less than 7 were classified.
(ii) The isomorphism of the basic polygroups was investigated.
(iii) The finite basic polygroup was fully described.

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