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APPROXIMATION OF SYMMETRICALLY RECIPROCAL MATRICES USING MUTATIONS IN MAX ALGEBRA

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Abstract. The objective of this paper is to propose a method for constructing a transitive matrix by maximizing the mutation of a symmetrically reciprocal matrix A, such that the resulting matrix is closest to A in terms of a relative error measure. By employing this approach, the need for calculating the maximum eigenvector is eliminated, leading to faster results. Additionally, we investigate the impact of mutations on two cases of change, specifically when a single measurement is corrected or when a new alternative is added, and analyse their effectiveness in ranking.

1. Introduction

An (entry wise) positive $n \times n$ matrix $A = (a_{ij})$ is called a symmetrically reciprocal matrix (SR-matrix) if $a_{ij}a_{ji} = 1$ for all $i, j = 1, \ldots, n$, thus $a_{ii} = 1$. These matrices first time were introduced by Saaty [4] and used in the analytic hierarchy process(AHP method) for multi criteria decision making. An SR- matrix $B = (b_{ij})$ is called transitive if there is a positive vector $w = (w_1, w_2, \ldots, w_n)$ such that $b_{ij} = \dfrac{w_i}{w_j}$ for $i, j = 1,\ dots, n$. Sometimes it is required to deduce positive weights $w_1, w_2, \ldots, w_n$ attached to the alternatives $A_1, \ldots, A_n$ respectively, from the SR-matrix A. These matrices can be

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ranked [1]. If $a_{ij} = \dfrac{w_i}{w_j}$ for $i, j = 1, 2, \dots, n$, then we can obtain ideal case, that is a transitive (consistent) matrix of rank one.

In [2] for constructing a weight vector $w = (w_1, w_2, \dots, w_n)$ using that of max-eigenvector of A that solves a useful optimization problem that is, minimizing the relative error useful optimization problem, namely minimizing relative error

$$e(w) = \max \left| \frac{a_{ik} - \dfrac{w_i}{w_j}}{a_{ik}} \right|. \tag{1.1}$$

This paper presents a ground breaking method for constructing a weight vector, denoted as $W = (w_1, w_2, \dots, w_n)$, by leveraging matrix mutations. Compared to existing methods, our approach offers several distinct advantages.

2. Max algebra

The max-algebra system is one of the analogues of linear algebra that has recently attracted the attention of many researchers. The *max-algebra system* consists of nonnegative real numbers $\mathbb{R}_+$ equipped with the operations of multiplication $a \otimes b = ab$, and maximization $a \oplus b = \max\{a, b\}$.

The identity element for $\oplus$ is 0 i.e., $a \oplus 0 = 0 \oplus a = a$ for all $a \in \mathbb{R}_+$. Operations of the max algebra $\oplus$ and $\otimes$ extend to vectors and matrices in the same way as in conventional linear algebra. For $A, B \in \mathbb{R}_+^{n\times n}$, we denote the sum by $(A \oplus B)_{ij} = \max(a_{ij}, b_{ij})$ and the product by $(A \otimes B)_{ij} = \max_{1\le k\le n}(a_{ik}b_{kj})$.

For an $n \times n$ non-negative matrix A, the eigenequation in the max algebra is given by

$$A \otimes x = \lambda x$$

that $\lambda \ge 0$ is a max eigenvalue and $x > 0$ is a max eigenvector.

Definition 2.1. [3] *Suppose that $A = [a_{ij}]$ is a nonnegative matrix. Each term in the determinant of A of the form $a_{1\sigma(1)} \cdot a_{2\sigma(2)} \cdot \dots \cdot a_{n\sigma(n)}$, is called a mutation. A mutation is called a major mutation if none of the elements in the mutation are from the main diagonal of A, and it is called a minor mutation if at least one element in the mutation is from the main diagonal of A. If all elements in a mutation are from the main diagonal of A, then it is referred to as a principal mutation.*

Example 2.2. [3] *Consider the following matrix*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

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We have

$$det\ A = 1\cdot 5\cdot 9 - 1\cdot 6\cdot 8 - 2\cdot 4\cdot 9 + 2\cdot 6\cdot 7 + 3\cdot 4\cdot 8 - 3\cdot 5\cdot 7\,,$$

where $3\cdot 4\cdot 8$ *and* $2\cdot 6\cdot 7$ *are major mutations,* $1\cdot 5\cdot 9$ *is a principal mutation, and other terms are minor mutations.*

3. Approximation of symmetrically reciprocal matrices

In this section, we will present several theorems regarding the relationship between mutations and the symmetrically reciprocal matrix, as well as the transitive matrix. We will also discuss two types of changes that can be made to mutations. The first change involves altering a pair in the inverse of the symmetrically reciprocal matrix, with the optimal change being applied to the main mutation that has the highest possible value, resulting in modifications to the ranking matrix. The second change is accomplished using mutations. In this approach the new alternatives are introduced, and based on the elements of the main mutation, a new row and column are added to the symmetrically reciprocal matrix in such a way that the ranking of the new matrix remains unaffected. Given the definition of the symmetric two-sided matrix and the definition of mutations, it is evident that the product of all the elements in a mutation of the symmetrically reciprocal matrix is greater than or equal to zero. Therefore, based on this observation we can state the following theorem.

Theorem 3.1. *A matrix A is transitive if and only if* $a_{ii} = 1$ *and the product of all the mutations of A is equal to 1.*

Theorem 3.2. [1] *Let* $A = (a_{ij}) \in M_n(\mathbb{R}+)$ *be a symmetrically reciprocal (SR) matrix, and* $w = (w_1, \cdots, w_n)^T$ *a positive vector and* $B = (b_{ij})$, *where* $b_{ij} = w_i/w_k$. *Let* $c > 0$. *Then the following statements are equivalent*

(a) $|a_{ij} - b_{ij}| \leq ca_{ij}, \qquad i, j = 1, 2, \cdots, n,$
(b) $a_{ij}\frac{w_j}{w_i} \leq 1 + c, \qquad i, j = 1, 2, \cdots, n.$

In particular, choosing $w_i = x_i, i = 1, ..., n,$ *where* $x = (x_1, \cdots, x_n)^T$ *is a maxeigenvector of A with max-eigenvalue* $\mu(A)$, *then* $c = \mu(A)1$ *is minimal, and*

$$\min_{w>0} e(w) = e(x) = \max_{i,j}\left|\frac{a_{ij} - \frac{x_i}{x_j}}{a_{ij}}\right| = \mu(A) - 1.$$

Let A be a positive symmetrically reciprocal (SR) matrix, and let $\tau > 0$. We define $M_{\max}$ as the set of all elements in the major mutation that corresponds to the maximum mutation, and $M_{\min}$ as the set of all elements in the major mutation that corresponds to the minimum mutation. Let $A(\tau) = [a_{ij}^{\tau}]$

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where

$$a_{ij}^{\tau} = \begin{cases} a_{sp}\tau, & a_{sp} \in M_{\max}; \\ \frac{a_{ps}}{\tau}, & a_{ps} \in M_{\min}; \\ a_{ij}, & \text{other wise.} \end{cases} \tag{3.1}$$

Lemma 3.3. *For $n \geq 3$, there exist positive numbers $T_1 \leq T_2$ such that the function $\mu(\tau)$ is strictly decreasing in the interval $(0, T_1]$, constant in the interval $[T_1, T_2]$, and strictly increasing in the interval $[T_2, \infty)$. In other words, the behaviour of the function $\mu(\tau)$ changes at T_1 and T_2, with a monotonicity pattern of decreasing, constant, and then increasing.*

Lemma 3.4. *Suppose A is a symmetrically reciprocal (SR) matrix, and $\mu(A)$ is obtained by the main mutation of length n. In this case, the matrix $A(\mu)$ is a transitive matrix.*

Theorem 3.5. *Let in (3.1), $(s,p) = (1,2)$, $0 < \tau_0 < \tau_1$ and x, y be two max eigenvector of $A(\tau)$ such that*

$$A(\tau_0) \otimes x = \mu(\tau_0)x \qquad A(\tau_1) \otimes y = \mu(\tau_1)y. \tag{3.2}$$

Then

1. *If $\mu(\tau_0) < \mu(\tau_1)$, then $\frac{y_1}{x_1} > \frac{y_i}{x_i}$,* $\quad i \geq 2$.
2. *If $\mu(\tau_0) > \mu(\tau_1)$ then $\frac{y_2}{x_2} < \frac{y_i}{x_i}$,* $\quad i \neq 2$.

Now, let's consider the scenario where a new alternative is added to the weight vector construction method. In this case, our method involves adding a new row and column to the existing matrix by utilizing mutation. We begin with an $n \times n$ symmetrically reciprocal (SR) matrix A.

Let $A \in M_n$ be an SR matrix with an eigenvalue μ and an eigenvector x, satisfying $A \otimes x = \mu x$. Additionally, let $s = (s_1, s_2, \ldots, s_n)$ be a vector such that $s^{-1} = (s_1^{-1}, s_2^{-1}, \ldots, s_n^{-1})$.

To accommodate the addition of a new alternative, we define a new matrix A_e as follows:

$$A_e = \begin{pmatrix} A & s \\ s^{-1} & 1 \end{pmatrix}. \tag{3.3}$$

Theorem 3.6. *Let A be a SR matrix with max-eigenvalue $\mu(A) > 1$. Then following are equivalent:*

1. *$\mu(A) = \mu(A_e)$ and there is $\alpha > 0$ such that (x^T, α) is the max-eigenvector of A_e.*
2. $\max_k(\frac{s_k}{x_k}) \max_k(\frac{x_k}{s_k}) \leq \mu^2$.

Sufficient condition for s to satisfy conditions (1) and (2) are: there exist $h > 0$ such that $s = A \otimes h$ or $s = Ah$.

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