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GENERATION OF SYMMETRICAL PATTERNS USING DISCRETE DYNAMICAL SYSTEM

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Abstract. Nowadays symmetrical patterns are widely used in various industries, such as jewelry design, carpet design, patterns on wallpaper, and textile design. During the design stage, designers perform most of the work manually. Therefore, the development of methods for symmetrical pattern generation is beneficial. In this paper, we are going to present some methods for generating symmetrical patterns using the discrete dynamical system. For this, the discrete dynamical system is considered as a standard iterative method, and then the general algorithm applied for Polynomiography is used for this to generate patterns. Through phase portrait, we analyze the conditions of the existence of some conventional symmetries. Several non-standard iterative methods can be employed to create a variety of visually appealing patterns. These methods include Mann iteration, Ishikawa iteration, and S-iteration which we use them. Through numerous examples, it is demonstrated that by manipulating the parameters and coefficients, it is possible to generate beautiful symmetrical patterns that have potential artistic applications.

1. Introduction

Symmetrical patterns have been used since ancient times in various industries, such as jewelry design, carpet design, and fabric design. Today, with the increasing use of computers, the generation of symmetrical patterns is carried out by computers and algorithms. In this work, we demonstrate how to utilize iterative processes, discrete dynamical systems, and the applied algorithm for Polynomiography

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to generate aesthetic and symmetrical patterns. To get to know Polynomiography, refer to [9, 10, 2, 1, 8, 3].

This paper is structured as follows: in section 2, we introduce some non-standard iteration processes and an algorithm to generate symmetrical patterns. In section 3, some basic information on generating symmetrical patterns from discrete dynamical systems is presented. We present various examples of symmetrical patterns obtained from the proposed technique in section 4. Finally, section 5 contains the conclusion.

2. Overview of iterative methods

We can express an iterative method as the general form

$$X_{n+1} = T(X_n),\ n = 0, 1, \dots. \tag{2.1}$$

In the above iterative method, the sequence $\{X_n\}_{n=0}^{\infty}$ is named the orbit of the starting point X_0. Our objective is to create patterns, so based on the method (2.1), we can formulate the following non-standard iterative techniques:

1. Mann iteration (which is a one-step iteration):

$$X_{n+1} = (1-\alpha)X_n + \alpha T(X_n), n = 0, 1, \dots. \tag{2.2}$$

2. Ishikawa iteration (which is a two-step iteration):

$$\begin{cases} X_{n+1} = (1-\alpha)X_n + \alpha T(V_n), \\ V_n = (1-\beta)X_n + \beta\ T(X_n),\ n = 0, 1, 2, \dots. \end{cases} \tag{2.3}$$

3. $S-$iteration (which is a two-step iteration):

$$\begin{cases} X_{n+1} = (1-\alpha)\ T(X_n) + \alpha\ T(V_n), \\ V_n = (1-\beta)X_n + \beta\ T(X_n),\ n = 0, 1, 2, \dots. \end{cases} \tag{2.4}$$

where $0 \leq \alpha \leq 1,\ 0 \leq \beta \leq 1$. Moreover, we notice that the iteration for particular values of the parameters can be reduced to other iterations. For instance, if we take $\alpha = 1,\ \beta = 0$, the above non-standard methods reduce to the standard method (2.1). Also, with $\beta = 0$, Ishikawa reduces to Mann. Unlike most works, in which constant parameters have been used, we can apply variable parameters and consider $\alpha = \alpha_n$ and $\beta = \beta_n$ [2]. A review of various nonstandard iterations and their dependencies can be found in [5].

To generate patterns, it is sufficient to apply an iterative method in the complex plane, it means $X_n = (x_n, y_n)^T$. We need a stop criterion or a convergence test in the iteration processes. In this work, we use the following standard convergence test

$$\|X_{n+1} - X_n\|_2 = \sqrt{(x_{n+1}-x_n)^2 + (y_{n+1}-y_n)^2} \leq \epsilon, \tag{2.5}$$

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where $\epsilon > 0$ is a given accuracy and X_n, X_{n+1} are two successive points in an iterative process. In [6, 7], Gdawice proposed some different convergence tests, for instance

$$(|x_{n+1} - x_n|^q + |y_{n+1} - y_n|^q)^{\frac{1}{q}} < \epsilon,$$
$$|(a|x_{n+1} - x_n|)^q - (b|y_{n+1} - y_n|)^r| < \epsilon, \ a, b \in \mathbb{R}, \ q, r \in \mathbb{R}_+.$$

Now, we can use the following algorithm to generate patterns [2, 1, 7]:

Algorithm 1: Iteration colouring.

Input: $A \subset \mathbb{C}$ − area, K − maximum number of iterations, I − iteration method, C − convergence test.

Output: a symmetrical pattern on the area A.

```
for X_0 ∈ A do
    i = 0
    while i ≤ K do
        X_{i+1} = I(X_i)
        if C(X_i, X_{i+1}) = true then
            break
        i = i + 1
    determine the colour for X_0 according to i.
```

3. symmetry in a discrete dynamical system

Dynamical systems are mathematical models that contain rules describing how a quantity changes over time. In this work, we consider the following discrete dynamical system[7, 4]

$$\begin{cases} x_{n+1} = x_n - f(x_n, y_n); \\ y_{n+1} = y_n - g(x_n, y_n). \end{cases} \tag{3.1}$$

where $(x_0, y_0)^T \in \mathbb{R}^2$ and $f, g : \mathbb{R}^2 \to \mathbb{R}$ are known functions. We can consider the above form as an iterative process and apply the defined algorithm for it. In what follows, we will explore certain conditions under which the resulting patterns exhibit a unique symmetry.

■ Translation Symmetry: Suppose that the pattern has a period T along the $x-$axis. Thus, the phase portrait must have this property and

$$\begin{cases} x_{n+1} + T = (x_n + T) - f(x_n + T, y_n), \\ y_{n+1} = y_n - g(x_n + T, y_n). \end{cases} \tag{3.2}$$

by comparing (3.1) and (3.2), we have

$$f(x_n+T,y_n)=f(x_n,y_n),\ \ g(x_n+T,y_n)=g(x_n,y_n). \tag{3.3}$$

Similarly, if the pattern has a period T^* along the $y-$axis, the following relation must be held:

$$f(x_n,y_n+T^*)=f(x_n,y_n),\ \ g(x_n,y_n+T^*)=g(x_n,y_n) \tag{3.4}$$

■ Reflective Symmetry: Suppose that the pattern has reflective symmetry about the $x-$axis. Thus, the phase portrait must have this property and

$$\begin{cases} x_{n+1}=x_n-f(x_n,-y_n), \\ -y_{n+1}=-y_n-g(x_n,-y_n). \end{cases} \tag{3.5}$$

by comparing (3.1) and (3.5), it results that in order to satisfy this property, the following relations must be held:

$$f(x,y)=f(x,-y),\ \ g(x,-y)=-g(x,y); \tag{3.6}$$

Similarly, if the pattern has reflective symmetry about the $y-$axis, then

$$\begin{cases} -x_{n+1}=-x_n-f(-x_n,y_n), \\ y_{n+1}=y_n-g(-x_n,y_n). \end{cases} \tag{3.7}$$

by comparing (3.1) and (3.7), it results that

$$f(-x,y)=-f(x,y),\ \ g(-x,y)=g(x,y). \tag{3.8}$$

Consequently, if we want to have reflective symmetry about both the $x-$axis and $y-$axis, it is sufficient to have the following properties:

$$\begin{cases} f(x,y)=f(x,-y)=-f(-x,y), \\ g(x,y)=-g(x,-y)=g(-x,y). \end{cases} \tag{3.9}$$

■ Glide Reflective Symmetry: One of the seen symmetries in nature and art is glide reflective symmetry. Suppose that the pattern has a period T along the $x-$axis and a glide reflection in the same direction. Therefore the phase portrait must have these properties and

$$f(x,y)=f(x+\frac{T}{2},-y),\ \ -g(x,y)=g(x+\frac{T}{2},-y). \tag{3.10}$$

Similarly, if we want these properties to satisfy along the $y-$axis, then

$$-f(x,y)=f(-x,y+\frac{T}{2}),\ \ g(x,y)=g(-x,y+\frac{T}{2}). \tag{3.11}$$

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■ Rotational Symmetry: If we want to obtain a pattern with a rotational symmetry with angle θ, then the functions $f,\ g$ should fulfill the following conditions [7, 4]:

$$f(x'',y'') - 2\cos(\theta) f(x',y') + f(x,y) = 0,\ \ g(x,y) = \frac{\cos(\theta)f(x,y) - f(x',y')}{\sin(\theta)}; \tag{3.12}$$

where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = R_\theta \begin{bmatrix} x \\ y \end{bmatrix},$$
$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = R_\theta \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

For example, in rotational symmetry with respect to $\theta = 90°$,the functions f and g should satisfy the following conditions:

$$f(x,y) = -f(-x,-y),\ \ f(x,y) = f(x+T,y) = f(x,y+T),\ \ g(x,y) = -f(-y,x). \tag{3.13}$$

To apply the non-standard iterative methods to generate a symmetrical pattern, it is sufficient to rewrite the discrete dynamical system in standard form $p_{n+1} = T(p_n)$, where

$$p_{n+1} = \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix},\ \ T(p_n) = \begin{bmatrix} x_n - f(x_n,y_n) \\ y_n - g(x_n,y_n) \end{bmatrix}. \tag{3.14}$$

Therefore, the Mann, Ishikawa and $S-$iteration are defined respectively as follows:

$$p_{n+1} = (1-\alpha_n)p_n + \alpha_n T(p_n), \tag{3.15}$$

$$\begin{cases} p_{n+1} = (1-\alpha_n)p_n + \alpha_n T(u_n) \\ u_n = (1-\beta_n)p_n + \beta_n T(p_n)\ , \end{cases} \tag{3.16}$$

$$\begin{cases} p_{n+1} = (1-\alpha_n)T(p_n) + \alpha_n T(u_n) \\ u_n = (1-\beta_n)p_n + \beta_n T(p_n)\ . \end{cases} \tag{3.17}$$

4. EXAMPLES

In this section, we perform a series of experiments to generate symmetrical patterns. Algorithm 1 has been implemented in Matlab. In all examples, we consider the convergence test

$$\sqrt{(x_{n+1}-x_n)^2 + (y_{n+1}-y_n)^2} \le 0.1, \tag{4.1}$$

and the maximum number of iterations $K = 50$. Fig (1) presents some patterns obtained using Algorithm 1. The functions used to generate these patterns have been listed as follows:

$$(A):\begin{cases} f(x,y)=\sin^3(x)\cos(y) \\ g(x,y)=\cos(2x)\sin^2(y), \end{cases} \tag{4.2}$$

$$(B):\begin{cases} f(x,y)=\sin(x)\cos(y) \\ g(x,y)=\cos(2x)\sin^3(\sqrt{5}y), \end{cases} \tag{4.3}$$

$$(C):\begin{cases} f(x,y)=0.4\sin(x)\cos(\sqrt{5}y)-0.2\sin(y)\cos(\sqrt{5}x) \\ g(x,y)=0.4\sin(y)\cos(\sqrt{5}x)+0.2\sin(x)\cos(\sqrt{5}y), \end{cases} \tag{4.4}$$

$$(D):\begin{cases} f(x,y)=0.4\sin(\sqrt{2}x)\cos(2y)+0.2\sin(3.2x)\cos(y) \\ g(x,y)=0.4\sin(y)\cos(2x)+0.2\sin(2y)\cos(x). \end{cases} \tag{4.5}$$

As seen, by varying of some the coefficients, the new different patterns are generated. Now, in order to use non-standard iterations and to investigate the efficiency of parameters, consider the following functions:

$$f(x,y)=0.4\sin(x)\cos(2y)-0.2\sin(x)\cos(2x)$$
$$g(x,y)=0.4\sin(y)\cos(2x)+0.2\sin(x)\cos(2y)$$

Fig (2) presents some generated patterns using the above functions. The applied parameters are shown in Table (1).

Table 1. The applied parameters to generat pattern.

pattern	parameter	iterative method
E	-	standard
F	$\alpha(n)=0.45$	Mann
G	$\alpha(n)=\cos(\sqrt{7}n), \beta(n)=\frac{n+1}{n}$	$S-$iteration
H	$\alpha(n)=\cos(\sqrt{5}n), \beta(n)=\frac{n^2+1}{n^2}$	$S-$iteration

5. Conclusions

In this article, we tried to produce symmetrical patterns by using the concept of convergence of the sequences generated by the discrete dynamic system. In order to generate attractive patterns, we also used the non-standard forms of iterative processes. We observed that with small changes, various patterns are produced. This method can be used in some industries such as making tiles, decorating clothes, etc. Also, these symmetrical patterns can be used in students' stationery, such as notebooks,

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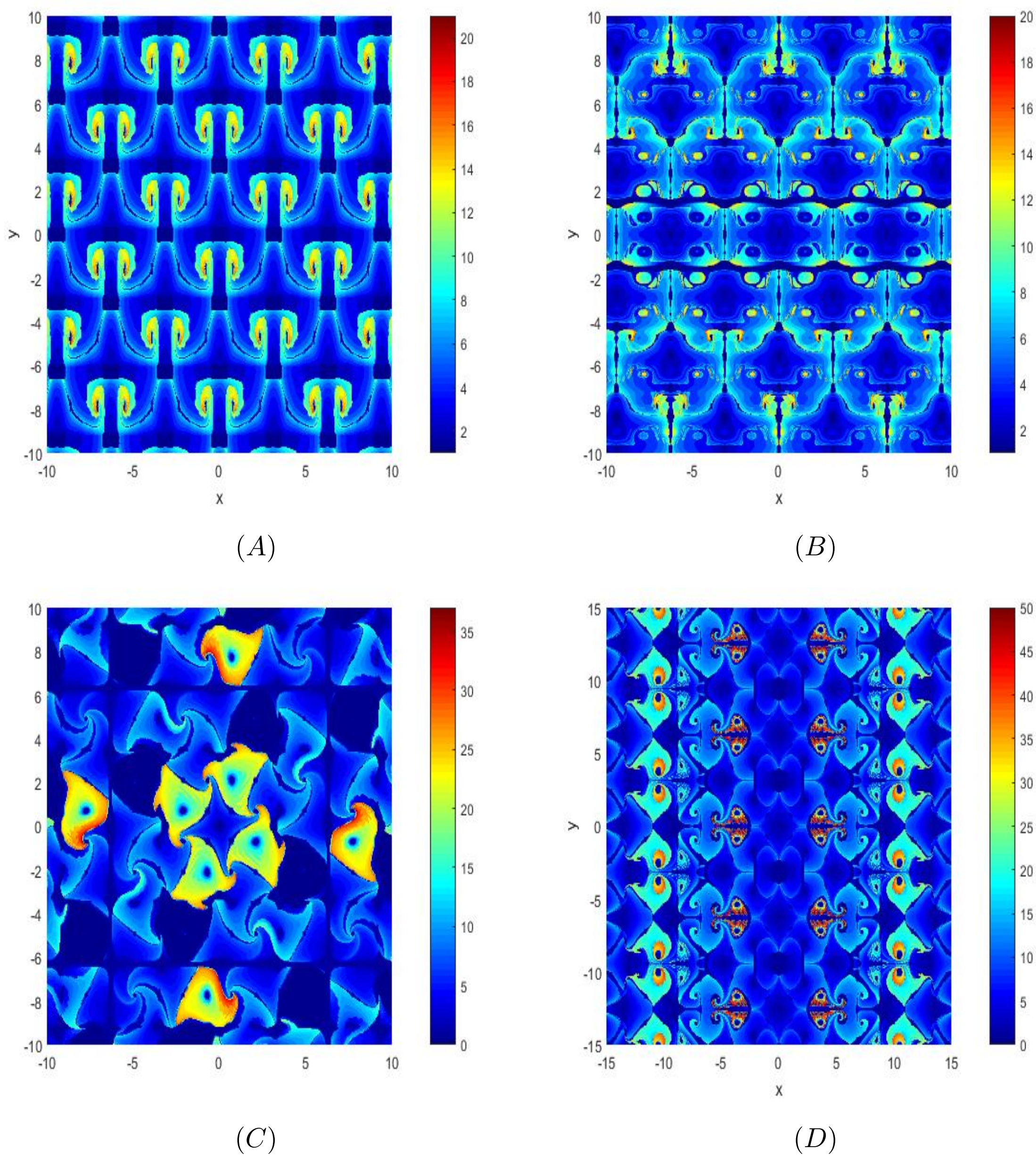
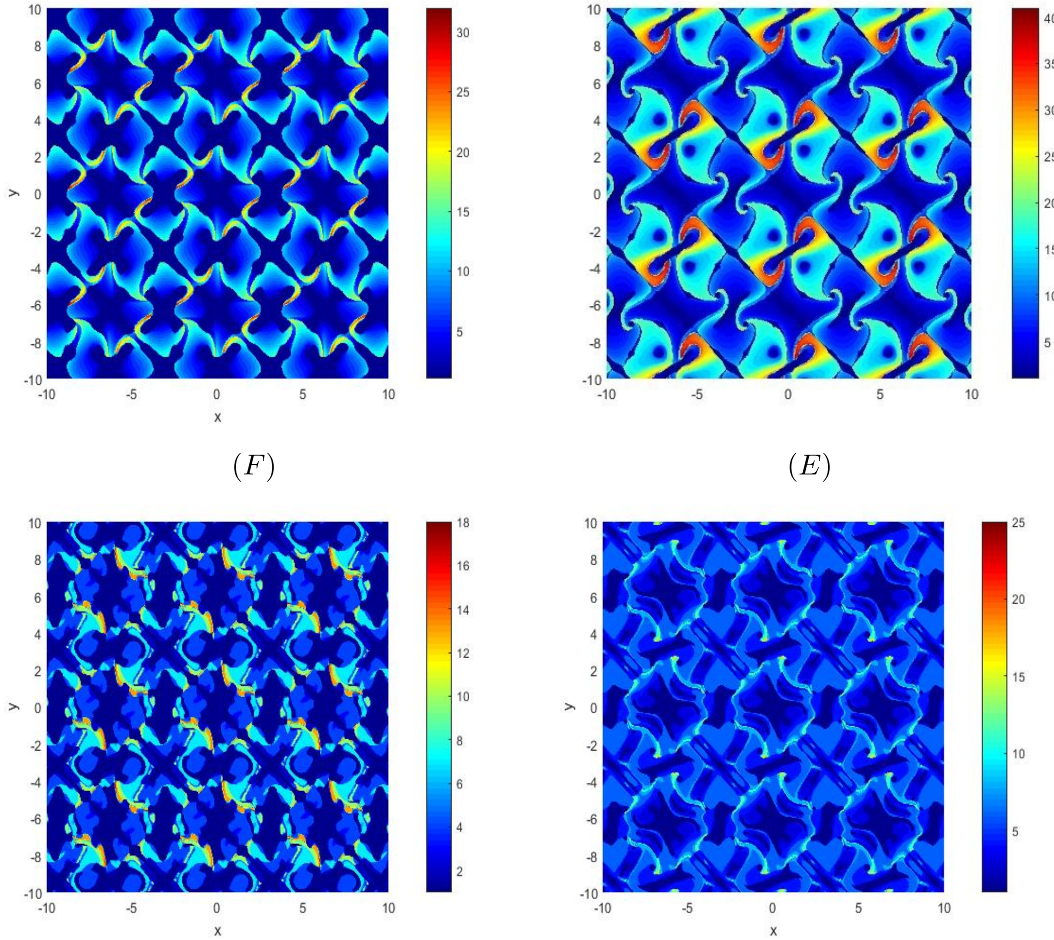


Figure 1. Examples of obtained patterns.



(F) (E)

(H) (G)

Figure 2. Examples of obtained patterns.

along with their simple and concise explanations. This causes an increase in students' interest in mathematical concepts.

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