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RESULTS ABOUT CROSSED POLYSQUARES

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Abstract. Crossed polysquares are defined by Dehghanizadeh, Davvaz and Alp. Their properties and the generalization of results from intersecting squares to intersecting polysquares have been expressed and proved by them with the help of fundamental relations. In the following, the concept of intersected polymodules and Γ-equivalent, intersected polymodules of polygroups are introduced and some properties are obtained from it. In addition, the concept of hypermultiplying fiber and crossed polysquares in the homotopy form of kernels has been studied. These results have extended the results related to intersecting squares to intersecting polysquares. In this article, homotopy crossed polysquares are studied as homotopies of cokernel, then the image of a crossed polymodule is considered and some results are proved that show the correspondence between crossed polymodules and crossed polysquares. In the continuation of the studies, we can check the results about the crossed 2-squares and then the crossed 2-polysquares. In addition, concepts about intersecting n-squares and intersecting n-polysquares can be expanded and studied.

1. Introduction

Crossed modules and their applications play an important role in group theory, homotopy theory, homology and cohomology of groups, algebra, and k-theory, etc. Crossed modules were originally defined by Whitehead [24] as a model for 2-types. Loday in [21] proposed a sequence equivalent to the crossed module sequence called the 1-group sequence. Norrie in [22] provided a good example of

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crossed modules in the form of a crossed module action. Conduché [11] defined a 2-crossed module as a model for 3-types. In his unpublished work, he established that there is an equivalence between the category of square modules of groups and the crossed 2-modules of groups.

In [6] Arvasi and Porter showed how to get from a simplicity algebra to a 2-crossed module of algebras and vice versa, and they clarified the relationship between simplicity algebras and intersecting squares. As an algebraic model of three types, the concept of crossed 2-modules was introduced by Conduché in [11] and these crossed 2-modules are equivalent to simplicity groups with a Moore twist. Moore complex is of length two. Crossed squares and quadratic modules are other connected algebraic modules of the third type, defined by Loday, Guin-Walery, and Baues [8], are defined respectively. In [7], Arvasi and Ulualan studied the relations between crossed 2-modules, quadratic modules, crossed squares, and simplicity groups and homotopy equivalence between these superstructures. For more tips on cross modules, you can refer to [1, 2, 3, 5, 9, 18]. In [17] Loday and Guin-Walery presented the idea of treating a crossed square as an algebraic model of connected 3. Crossed polysquares studied by Dehghanizadeh, Davvaz, and Alp in [14, 15].

2. Main Results

Definition 2.1. *A crossed polysquares is a commutative diagram of polygroups*

$$\begin{array}{ccc} P_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ P_0 & \xrightarrow[\bar{p}_0]{} & \Gamma_0 \end{array}$$

diagram (1)

together with polyactions of the polygroup Γ_0 on P_1, Γ_1 and P_0 (and hence polyactions of Γ_1 on P_1 and P_0 via ∂' and of P_0 on P_1 and Γ_1 via $\bar{p}_0$) and a function $h : \Gamma_1 \times P_0 \longrightarrow \mathcal{P}^*(P_1)$, such that the following axioms are satiesfied:

(1) *the maps $\bar{p}_1, \partial$ preserve the polyactions of Γ_0. Furthermore, with the given polyactions the maps ∂', $\bar{p}_0$ and $\partial'\bar{p}_1 = \bar{p}_0\partial$ are crossed polymodules;*
(2) $\bar{p}_1 h(\beta, p) = \beta^p\beta^{-1}$, $\partial h(\beta, p) = {}^{\beta}pp^{-1}$;
(3) $h(\bar{p}_1(\alpha), p) = \alpha\,{}^{p}\alpha^{-1}$, $h(\beta, \partial(\alpha)) = {}^{\beta}\alpha\alpha^{-1}$;
(4) $h(\beta_1\beta_2, p) = {}^{\beta_1}h(\beta_2, p)h(\beta_1, p)$, $h(\beta, p_1p_2) = h(\beta, p_1)\,{}^{p_1}h(\beta, p_2)$;
(5) $h({}^{\sigma}\beta, {}^{\sigma}p) = {}^{\sigma}h(\beta, p)$;

for all $\alpha \in P_1$, $\beta, \beta_1, \beta_2 \in \Gamma_1$, $p, p_1, p_2 \in P_0$ and $\sigma \in \Gamma_0$.

Theorem 2.2. *Let diagram (1) be a crossed polysquare, then outer diagram*

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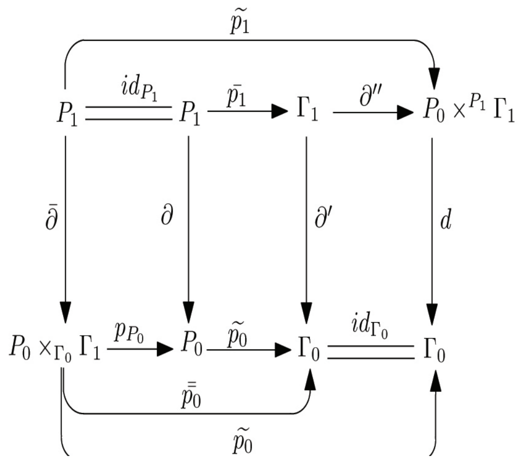

diagram (2)

gives rise to a crossed polysquare with actions, polygroup homomorphism ∂'' *and function* $\bar{\bar{h}} : (P_0 \ltimes {}^{P_1}\Gamma_1) \times (P_0 \times {}_{\Gamma_0}\Gamma_1) \longrightarrow \mathcal{P}^*(P_1)$ *defined as following:*

(i) *the polyacrion of* Γ_0 *on* P_1 *is induced by the polyaction of crossed polymodule of* $\partial' : \Gamma_1 \longrightarrow \Gamma_0$ *on* $\partial : P_1 \longrightarrow P_0$*;*

(ii) *the polyaction of* Γ_0 *on* $P_0 \ltimes {}^{P_1}\Gamma_1$ *is the polyaction of a crossed polymodule* $d : P_0 \ltimes {}^{P_1}\Gamma_1 \longrightarrow \Gamma_0$*;*

(iii) *the polyaction of* Γ_0 *on* $P_0 \times_{\Gamma_0} \Gamma_1$ *is defined by* ${}^{\sigma}(p_2, \beta_2) = \{(x, y) \mid x \in {}^{\sigma}p_2,\ y \in {}^{\sigma}\beta_2\}$ *(the same polyaction seen in the crossed polysquare diagram (2);*

(iv) $\partial'' : \Gamma_1 \longrightarrow P_0 \ltimes {}^{P_1}\Gamma_1$ *is the canonical inclusion map of* Γ_1 *in* $P_0 \ltimes {}^{P_1}\Gamma_1$*;*

(v) $\bar{\bar{h}}\,((p_1, \beta_1), (p_2, \beta_2)) := \left\{h(\beta_1, p_1 p_2 p_1^{-1}) h(\beta_2, p_1)^{-1}\right\}$ *where the function* h *is given by the crossed polysquare structure of diagram(1).*

Theorem 2.3. *If diagram (1) is a crossed polysquare, then the outer diagram is a crossed square with actions and function,*

$$\bar{h} : \frac{P_0 \ltimes^{P_1} \Gamma_1}{\beta^*_{P_0 \ltimes^{P_1}\Gamma_1}} \times \frac{P_0 \times_{\Gamma_0} \Gamma_1}{\beta^*_{P_0 \times_{\Gamma_0}\Gamma_1}} \longrightarrow \frac{P_1}{\beta^*_{P_1}}$$

defined as following

(a) *the action of* $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$ *on* $\frac{P_1}{\beta^*_{P_1}}$ *is induced by the polyaction of* Γ_0 *on* P_1*;*

(b) *the action of* $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$ *on* $\frac{P_0 \ltimes^{P_1}\Gamma_1}{\beta^*_{P_0 \ltimes^{P_1}\Gamma_1}} \ltimes^{P_1} \Gamma_1$ *by the polyaction of* Γ_0 *on* $P_0 \ltimes^{P_1} \Gamma_1$*;*

(c) *the action of* $\frac{\Gamma_0}{\beta^*_{\Gamma_0}}$ *on* $\frac{P_0 \times_{\Gamma_0}\Gamma_1}{\beta^*_{P_0 \times_{\Gamma_0}\Gamma_1}}$ *is induced by the polyaction of* Γ_0 *on* $P_0 \times_{\Gamma_0} \Gamma_1$*;*

(d) *the map* $\bar{h} : \frac{P_0 \ltimes^{P_1}\Gamma_1}{\beta^*_{P_0 \ltimes^{P_1}\Gamma_1}} \times \frac{P_0 \times_{\Gamma_0}\Gamma_1}{\beta^*_{P_0 \times_{\Gamma_0}\Gamma_1}} \longrightarrow \frac{P_1}{\beta^*_{P_1}}$ *is*

$$\bar{h}((\beta^*_{P_0}(p_0), \beta^*_{\Gamma_1}(\gamma_1), (\beta^*_{P_0}(p_0'), \beta^*_{\Gamma_1}(\gamma_1')) = \beta^*_{P_1}(h((p_0, \gamma_1), (p_0', \gamma_1'))).$$

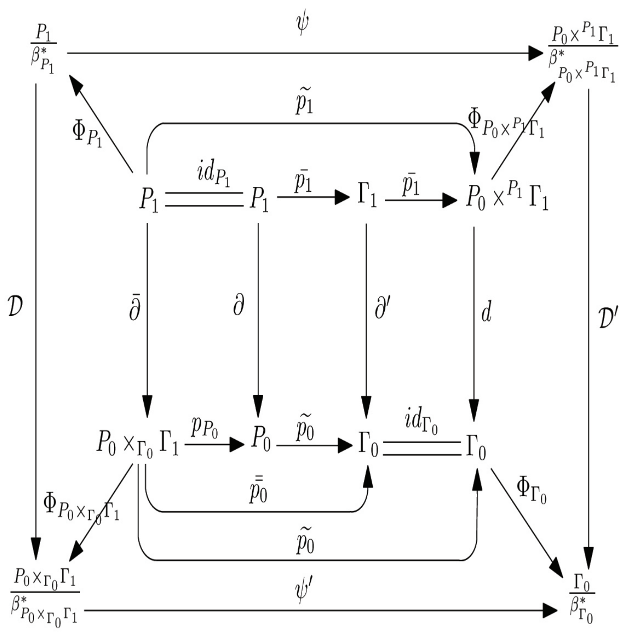

diagram (3)

3. Summary of Proofs

(Summary of Proof 2-2). $\tilde{p}_0 = \bar{\bar{p}}_0$ is a strong homomorphism, where $\bar{\bar{p}}_0$ is defined in diagram. $\bar{p}_1$ is a strong homomorphism, so $\tilde{p}_1(\alpha) = (1, \bar{p}_1(\alpha))$ is a strong homomorphism. The diagram is commutes and the last map is a crossed polymodule, because it is easy to check that $d\tilde{p}_1 = \partial'\bar{p}_1 = \bar{p}_0\partial = \tilde{p}_0\bar{\partial}$. But $\bar{\bar{h}}$ is well defined, in fact we have:

$$
\begin{aligned}
&\bar{\bar{h}}\left\{((x,y),(p_2,\beta_2)) \mid x \in \partial(\alpha)p_1,\, y \in \beta_1\bar{p}_1(\alpha)^{-1}\right\} \\
&\quad = h\left\{(x,y) \mid x \in \beta_1\bar{p}_1(\alpha)^{-1},\, y \in \partial(\alpha)p_1p_2p_1^{-1}\partial(\alpha)\right\}\, h\left\{(\beta_2, z)^{-1} \mid z \in \partial(\alpha)p_1\right\} \\
&\quad = h\left\{(x,y) \mid x \in \beta_1\bar{p}_1(\alpha)^{-1},\, y \in {}^{\bar{p}_0\partial(\alpha)}(p_1p_2p_1^{-1})\right\}\, {}^{\partial(\alpha)}h(\beta_2,p_1)^{-1}h\left\{(\beta_2,z)^{-1} \mid z \in \partial(\alpha)\right\} \\
&\quad = {}^{\bar{p}_0\partial(\alpha)}h\left\{(x,y) \mid x \in {}^{\bar{p}_0\partial(\alpha)^{-1}}(\beta_1\bar{p}_1(\alpha)^{-1},\, y \in p_1p_2p_1^{-1})\right\}\, \alpha h\left\{(\beta_2,p_1)^{-1}\,\alpha^{-1}\alpha^{\beta_2}\alpha^{-1}\right\} \\
&\quad = \alpha h\left\{(x,y)\ \ x \in {}^{\partial'\bar{p}_1(\alpha)^{-1}}(\beta_1\bar{p}_1(\alpha)^{-1},\, y \in p_1p_2p_1^{-1}\right\}\alpha^{-1}\alpha h(\beta_2,p_1)^{-1\,\beta_2}\alpha^{-1} \\
&\quad = \alpha^{\partial'\bar{p}_1(\alpha)^{-1}}h\left\{(\beta_1,y) \mid y \in p_1p_2p_1^{-1}\right\} h\left\{(\bar{p}_1(\alpha)^{-1},y) \mid y \in p_1p_2p_1^{-1}\right\} h(\beta_2,p_1)^{-1\,\beta_2}\alpha^{-1} \\
&\quad = \alpha^{\partial'\bar{p}_1(\alpha)^{-1}}h\left\{(\beta_1,y) \mid y \in p_1p_2p_1^{-1}\right\} h\left\{(\bar{p}_1(\alpha)^{-1},y) \mid y \in p_1p_2p_1^{-1}\right\} h(\beta_2,p_1)^{-1\,\beta_2}\alpha^{-1} \\
&\quad = \alpha\alpha^{-1}h\left\{(\beta_1,y) \mid y \in p_1p_2p_1^{-1}\right\}\alpha\alpha^{-1\;p_1p_2p_1^{-1}}\alpha h(\beta,p_1)^{-1\,\beta_2}\alpha^{-1} \\
&\quad = h\left\{(\beta_1,y) \mid y \in p_1p_2p_1^{-1}\right\}{}^{p_1\beta_2}({}^{p_1^{-1}}\alpha)h(\beta_2,p_1)^{-1\,\beta_2}\alpha^{-1}
\end{aligned}
$$

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$$= h\left\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\right\} h(\beta_2, p_1)^{-1}\, {}^{\beta_2 p_1}({}^{p_1^{-1}}\alpha)\, {}^{\beta_2}\alpha^{-1}$$
$$= h\left\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\right\} h(\beta_2, p_1)^{-1}\, {}^{\beta_2}\alpha\, {}^{\beta_2}\alpha^{-1}$$
$$= h\left\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\right\} h(\beta_2, p_1)^{-1}.$$

Outer diagram is crossed polysquare and so the equalities above consequences of the axioms of the crossed polysquare.

Now we want to check the five properties making the diagram a crossed polysquare.

(i) the map $\tilde{p}_1$ preserves the polyactions of Γ_0; in fact

$$\tilde{p}_1({}^{\sigma}\alpha) = \{(1, x) \mid x \in \bar{p}_1({}^{\sigma}\alpha)\} = \{(1, x) \mid x \in {}^{\sigma}\bar{p}_1(\alpha)\} = {}^{\sigma}(1, \bar{p}_1(\alpha)) = {}^{\sigma}\tilde{p}_1(\alpha).$$

The map $\bar{\partial}$ preserves the polyactions of Γ_0. Also d is a crossed polymodule and $\tilde{p}_0$ is a crossed polymodule because $\bar{\bar{p}}_0$ is.

(ii) we want to prove that

$$\tilde{p}_1\left(\bar{\bar{h}}((p_1, \beta_1), (p_2, \beta_2))\right) = (p_1, \beta_1)^{(p_2, \beta_2)}(p_1, \beta_1)^{-1}$$

and we develop the two members separately:

$$\begin{aligned} \tilde{p}_1\left(\bar{\bar{h}}((p_1, \beta_1), (p_2, \beta_2))\right) &= \tilde{p}_1\left(h\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\}\, h(\beta_2, p_1)^{-1}\right) \\ &= \left(1, \bar{p}_1(h\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\}\, h(\beta_2, p_1)^{-1}\right) \\ &= \{(1, y) \mid y \in \beta_1\, {}^{p_1p_2p_1^{-1}}\beta_1^{-1}\, {}^{p_1}\beta_2\beta_2^{-1}\}; \end{aligned}$$

and

$$(p_1, \beta_1)^{(p_2, \beta_2)}(p_1, \beta_1)^{-1}$$

$$\begin{aligned} &= (p_1, \beta_1)^{\tilde{p}_0(p_2, \beta_2)}(p_1, \beta_1)^{-1} \\ &= (p_1, \beta_1)^{\tilde{p}_0(p_2)}(p_1^{-1}, {}^{p_1^{-1}}\beta_1^{-1}) \\ &= (p_1, \beta_1)\{(x, y) \mid x \in p_2p_1^{-1}p_2^{-1},\, y \in {}^{p_2p_1^{-1}}\beta_1^{-1}\} \\ &= \{(u, v) \mid u \in p_1p_2p_1^{-1}p_2^{-1},\, v \in \beta_1\, {}^{p_1p_2p_1^{-1}}\beta_1^{-1}\} \\ &= \{(r, s) \mid r \in \partial h(\beta_2, p_1)^{-1}1,\, s \in \beta_1\, {}^{p_1p_2p_1^{-1}}\beta_1^{-1}\, {}^{p_1}\beta_2\beta_2^{-1}\bar{p}_1 h(\beta_2, p_1)\}. \end{aligned}$$

Now we want to prove that

$$\bar{\partial}\bar{\bar{h}}((p_1, \beta_1), (p_2, \beta_2)) = {}^{(p_1, \beta_1)}(p_2, \beta_2)(p_2, \beta_2)^{-1};$$

and we develop the two members separately:

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$$
\begin{aligned}
&\bar{\partial}\bar{\bar{h}}((p_1,\beta_1),(p_2,\beta_2))\\
&= \bar{\partial}(h\{(\beta_1,y)\mid y\in p_1p_2p_1^{-1}\}h(\beta_2,p_1)^{-1})\\
&= \{(\partial h(\beta_1,y)\,\partial h(\beta_2,p_1)^{-1},\bar{p}_1h(\beta_1,y)\bar{p}_1h(\beta_2,p_1)^{-1})\mid y\in p_1p_2p_1^{-1}\}\\
&= \{(u,v)\mid u\in {}^{\beta_1}(p_1p_2p_1)^{-1}p_1p_2^{-1}p_1^{-1}p_1\,{}^{\beta_2}p_1^{-1},\ v\in \beta_1\,{}^{p_1p_2p_1^{-1}}\beta_1^{-1}\,{}^{p_1}\beta_2\beta_2^{-1}\}\\
&= \{(u,v)\mid u\in {}^{\beta_1}(p_1p_2p_1^{-1})p_1p_2^{-1}\,{}^{\bar{p}_0(p_2)}p_1^{-1},\ v\in \beta_1\,{}^{p_1}(\,{}^{\partial'(\beta_2)}(\,{}^{p_1^{-1}}\beta_1^{-1}))\,{}^{p_1}\beta_2\beta_2^{-1}\}\\
&= \{(u,v)\mid u\in {}^{\beta_1}(p_1p_2p_1^{-1})p_1p_2^{-1}p_2p_1^{-1}p_2^{-1},\ v\in \beta_1\,{}^{p_1}\beta_2\beta_1^{-1}\,{}^{p_1}\beta_2^{-1}\,{}^{p_1}\beta_2\beta_2^{-1}\}\\
&= \{(u,v)\mid u\in {}^{\beta_1}(p_1p_2p_1^{-1})p_2^{-1},\ v\in \beta_1\,{}^{p_1}\beta_2\beta_1^{-1}\beta_2^{-1}\};
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{(p_1,\beta_1)}(p_2,\beta_2)(p_2,\beta_2)^{-1} &= {}^{\partial'(\beta_1)\bar{p}_0(p_1)}(p_2,\beta_2)(p_2^{-1},\beta_2^{-1})\\
&= \{(u,v)\mid u\in {}^{\beta_1}(p_1p_2p_1^{-1}),\ v\in \beta_1\,{}^{g_1}\beta_2\beta_1^{-1}\}(p_2^{-1},\beta_2^{-1})\\
&= \{(u,v)\mid u\in {}^{\beta_1}(p_1p_2p_1^{-1})p_2^{-1},\ v\in \beta_1\,{}^{p_1}\beta_2\beta_1^{-1}\beta_2^{-1}\}.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
\bar{\bar{h}}(\tilde{p}_1(\alpha),(p_2,\beta_2)) &= \bar{\bar{h}}((1,\bar{p}_1(\alpha)),(p_2,\beta_2))\\
&= h(\bar{p}_1(\alpha),p_2)h(\beta_2,1)^{-1}=\alpha\,{}^{p_2}\alpha^{-1}=\alpha\,{}^{\bar{p}_0(p_2)}\alpha^{-1}\\
&= \alpha^{\bar{p}_0(p_2,\beta_2)}\alpha^{-1}=\alpha\,{}^{(p_2,\beta_2)}\alpha^{-1};
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\bar{h}}((p_1,\beta_1),\bar{\partial}(\alpha)) &= \bar{\bar{h}}((p_1,\beta_1),(\partial(\alpha),\bar{p}_1(\alpha)))\\
&= h\{(\beta_1,y)\mid y\in p_1\partial(\alpha)p_1^{-1}\}\,h(\bar{p}_1(\alpha),p_1)^{-1}\\
&= h\{(\beta_1,y)\mid y\in \partial(\,{}^{p_1}\alpha)\}\,h(\bar{p}_1(\alpha),p_1)^{-1}\\
&= {}^{\beta_1}(\,{}^{p_1}\alpha)\,{}^{p_1}\alpha^{-1}\,{}^{p_1}\alpha\alpha^{-1}={}^{\beta_1}(\,{}^{p_1}\alpha)\alpha^{-1}\\
&= {}^{\partial'(\beta_1)\bar{p}_0(p_1)}\alpha\alpha^{-1}={}^{d(p_1,\beta_1)}\alpha\alpha^{-1}={}^{(p_1,\beta_1)}\alpha\alpha^{-1}.
\end{aligned}
$$

(iv) we want to prove that:

$$
\bar{\bar{h}}((p_1,\beta_1)(p_1',\beta_1'),(p_2,\beta_2))={}^{(p_1,\beta_1)}\bar{\bar{h}}((p_1',\beta_1'),(p_2,\beta_2))\,\bar{\bar{h}}((p_1,\beta_1),(p_2,\beta_2))
$$

and we develop the two members separately:

$$\bar{\bar{h}}((p_1,\beta_1)(p_1',\beta_1'),(p_2,\beta_2))$$

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$$
\begin{aligned}
&= \bar{\bar{h}}\{((x,y),(p_2,\beta_2)) \mid x \in p_1p_1',\ y \in \beta_1{}^{p_1}\beta_1'\}\\
&= h\{(y,z) \mid y \in \beta_1{}^{p_1}\beta_1',\ z \in p_1p_1'p_2{p_2'}^{-1}p_1^{-1}\}\, h\{(\beta_2,r)^{-1} \mid r \in p_1p_1'\}\\
&= {}^{\beta_1}h\{(s,z) \mid s \in {}^{p_1}\beta_1',\ z \in p_1p_1'p_2{p_1'}^{-1}p_1^{-1}\}\, h\{(\beta_1,z) \mid z \in p_1p_1'p_2{p_1'}^{-1}p_1^{-1}\}\\
&= {}^{\beta_1}h(\beta_2,p_1')^{-1}h(\beta_2,p_1)^{-1}\\
&= {}^{\beta_1p_1}h\{(\beta_1',t) \mid t \in p_1'p_2{p_1'}^{-1}\}\, h\{(\beta_1,u) \mid u \in p_1{p_1'}^{\bar{p}_0(p_2)}(p_1p_1')^{-1}p_2\}\\
&= {}^{p_1}h(\beta_2,p_1')^{-1}h(\beta_2,p_1)^{-1}\\
&= {}^{\beta_1p_1}h\{(\beta_1',t) \mid t \in p_1'p_2{p_1'}^{-1}\}\, h\{(\beta_1,\partial h(\beta_2,r)^{-1}p_2 \mid r \in p_1p_1^{-1}\}\\
&\qquad {}^{p_1}h(\beta_2,p_1')^{-1}h(\beta_2,p_1)^{-1}\\
&= {}^{\beta_1p_1}h\{(\beta_1',t) \mid t \in p_1'p_2{p_1'}^{-1}\}\, {}^{\beta_1}h\{(\beta_2,r)^{-1} \mid r \in p_1p_1'\}\\
&\qquad h(\beta_1,p_2)h\{(\beta_2,r) \mid r \in p_1p_1^{-1}\}\, {}^{p_1}h(\beta_2,p_1')^{-1}h(\beta_2,p_1)^{-1}\\
&= {}^{\beta_1p_1}h\{(\beta_1',t) \mid t \in p_1'p_2{p_1'}^{-1}\}\, {}^{\beta_1}h\{(\beta_2,r)^{-1} \mid r \in p_1p_1^{-1}\}\\
&\qquad h(\beta_1,p_2)h(\beta_2,p_1)\, {}^{p_1}h(\beta_2,p_1')^{-1}h(\beta_2,p_1)^{-1}\\
&= {}^{\beta_1p_1}h\{(\beta_1',t) \mid t \in p_1'p_2{p_1'}^{-1}\}\, {}^{\beta_1}h\{(\beta_2,r)^{-1} \mid r \in p_1p_1^{-1}\}\, h(\beta_1,p_2);
\end{aligned}
$$

and

$$
\begin{aligned}
&{}^{(p_1,\beta_1)}\bar{\bar{h}}((p_1',\beta_1'),(p_2,\beta_2))\,\bar{\bar{h}}((p_1,\beta_1),(p_2,\beta_2))\\
&={}^{\beta_1p_1}\,[h\{(\beta_1',t) \mid t \in p_1'p_2{p_1'}^{-1}\}\, h(\beta_2,p_1')^{-1}]h\{(\beta_1,y) \mid y \in p_1p_2p_1^{-1}\}\, h(\beta_2,p_1)^{-1}\\
&={}^{\beta_1p_1}\ h\{(\beta_1',t) \mid t \in p_1'p_2{p_1'}^{-1}\}\, {}^{\beta_1p_1}h(\beta_2,p_1')^{-1}h\{(\beta_1,w) \mid w \in p_1{}^{\bar{p}_0(p_2)}p_1^{-1}p_2\}\\
&\quad h(\beta_2,p_1)^{-1}\\
&={}^{\beta_1p_1}\ h\{(\beta_1',t) \mid t \in p_1'p_2{p_1'}^{-1}\}\, {}^{\beta_1p_1}h(\beta_2,p_1')^{-1}h\{(\beta_1,w_1) \mid w_1 \in p_1{}^{\partial'(\beta_2)}p_1^{-1}p_2\}\\
&\quad h(\beta_2,p_1)^{-1}\\
&={}^{\beta_1p_1}\ h\{(\beta_1',t) \mid t \in p_1'p_2{p_1'}^{-1}\}\, {}^{\beta_1p_1}h(\beta_2,p_1')^{-1}h\{(\beta_1,w_2) \mid w_2 \in \partial h(\beta_2,p_1)^{-1}p_2\}\\
&\quad h(\beta_2,p_1)^{-1}\\
&={}^{\beta_1p_1}\ h\{(\beta_1',t) \mid t \in p_1'p_2{p_1'}^{-1}\}\, {}^{\beta_1p_1}h(\beta_2,p_1')^{-1}\, {}^{\beta_1}h(\beta_2,p_1)^{-1}h(\beta_2,p_1)^{-1}h(\beta_1,p_2)\\
&\quad h(\beta_2,p_1)\,h(\beta_2,p_1)^{-1}\\
&={}^{\beta_1p_1}\ h\{(\beta_1',t) \mid t \in p_1'p_2{p_1'}^{-1}\}\, {}^{\beta_1}[{}^{p_1}h(\beta_2,p_1')^{-1}h(\beta_2,p_1)^{-1}]h(\beta_1,p_2)\\
&={}^{\beta_1p_1}\ h\{(\beta_1',t) \mid t \in p_1'p_2{p_1'}^{-1}\}\, {}^{\beta_1}h\{(\beta_2,r)^{-1} \mid r \in p_1p_1'\}\, h(\beta_1,p_2).
\end{aligned}
$$

(v)

$$\begin{aligned}
\bar{\bar{h}}({}^{\sigma}(p_1,\beta_1), {}^{\sigma}(p_2,\beta_2)) &= \bar{\bar{h}}(({}^{\sigma}p_1, {}^{\sigma}\beta_1), ({}^{\sigma}p_2, {}^{\sigma}\beta_2)) \\
&= h\{({}^{\sigma}\beta_1, x) \mid x \in {}^{\sigma}p_1 {}^{\sigma}p_2 {}^{\sigma}p_1^{-1}\}\, h({}^{\sigma}\beta_2, {}^{\sigma}p_1)^{-1} \\
&= h\{({}^{\sigma}\beta_1, x) \mid x \in {}^{\sigma}(p_1p_2p_1^{-1})\}\, h({}^{\sigma}\beta_2, {}^{\sigma}p_1)^{-1} \\
&= {}^{\sigma}h\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\}\, {}^{\sigma}h(\beta_2, p_1)^{-1} \\
&= {}^{\sigma}[h\{(\beta_1, y) \mid y \in p_1p_2p_1^{-1}\}\, h(\beta_2, p_1)^{-1}] \\
&= {}^{\sigma}\bar{\bar{h}}((p_1,\beta_1),(p_2,\beta_2).
\end{aligned}$$

□

Remark 3.1. *The proof of the rest of the theorems is straightforward but lengthy.*

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