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CUBIC SEMISYMMETRIC GRAPHS OF ORDER $40p$

MOHAMMAD REZA SALARIAN AND JAVANSHIR REZAEI*

Abstract. A simple graph Γ is called semisymmetric if it is regular and edge-transitive but not vertex-transitive. A simple graph Γ is called cubic whenever it is 3-regular. An important research problem is the classification of semisymmetric cubic graphs of different orders. The purpose of this article is the classification of semisymmetric cubic graphs of order $40p$, where p is a prime number. We show that for $p \neq 3, 31$ such a graph Does not exist. Suppose p is a prime number, Folkman showed in[13] that there is no semisymmetric graph of order $2p$ or $2p^2$. We show that if Γ is a semisymmetric cubic graph of order $40p$, then $p = 3$ and Γ is isomorphic to a semisymmetric cubic graph of order 120 or $p = 31$ and Γ is isomorphic to the coset graph $C(L_2(31) : \mathbb{S}_4, \mathbb{S}_4)$. Our basic tools in this research are automorphism of graphs, simple groups, solevable groups and permutation groups.

1. Introduction

Throughout of this paper all graphs are finite, undirected and simple, i.e. without loops or multiple edges. A graph is said semisymmetric if it is regular and edge-transitive, but not vertex-transitive. An interesting research question is to classify connected cubic semisymmetric graphs of various types of orders. Folkman proved in [13] that there is no connected cubic semisymmetric graph of order $2p$ or $2p^2$. cubic semisymmetric graphs of orders $4p^2, 8p^2$ and $10p^3$ have respectively been classified in [2],[1] and [3]. The authors of these articles have shown that there is no cubic semisymmetric graphs

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*Corresponding author.

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of these orders. According to [23], only cubic semisymmetric graph is obtained for $p = 3$ and it is gray graph with 54 vertices. In [18],it has been shown that there is no cubic semisymmetric graph of order $4p^3$ and only cubic semisymmetric graph of order $8p^3$ is obtained $p = 3$. cubic semisymmetric graphs of orders $6p^2, 10p^2, 14p^2, 20p^2, 20p, , 6p^3, 12p^3, 34p^3$ have respectively been classified in [16],[19],[4],[11],[29],[23],[12],[18],[5],[10]. In this paper we will characterize connected cubic semisymmetric graphs of order $40p$. In fact we prove that if Γ is a connected cubic semisymmetric graph of order $40p$, p prime, then either $p = 3$ and Γ is isomorphic to the cubic semisymmetric graph of order 120 in or $p = 31$ and $\Gamma \cong C(L_2(31); S_4, S_4)$. So precisely, we shall prove the following theorem.

Theorem 1.1. *Let p be a prime and $p \neq 3, 31$. Then there is no connected cubic semisymmetric graph of order $40p$.*

2. Main Results

In this section, we prove theorem 1.1 First we state some lemmas and prove some of them.

Notation and Assumptions: According to [7], for $p = 3$ there is only one cubic semisymmetric graph of order 120 and for $p = 5, 7, 11, 13$ and 17 there is no cubic semisymmetric graph of order $40p$. Thus we can assume that $p > 19$. In the remaining of this paper Γ is a cubic connected semisymmetric graph of order $40p$, where $p > 19$ is a prime. Set $A = \mathrm{Aut}(\Gamma)$.

Lemma 2.1. *If $O_p(A) = 1$, then A does not have normal subgroup of orders 10 and 20.*

Proof. Let $\{U, V\}$ be the bipartition of Γ and $u \in U$. Then $|U| = |W| = 20p$ and $|A_u| = 2^r.3$, for some r, $0 \leq r \leq 7$. So $|A| = |A_u||U| = 2^{r+2}.3.5.p$. Assume that $O_p(A) = 1$ and let M be a normal subgroup of A of order 20. Then M is intransitive on U and , Γ_M is a $\frac{A}{M}$-semisymmetric graph with bipartition $\{U_M, W_M\}$. We note that $|\frac{A}{M}| = 2^r.3.p$, and $|U_M| = |W_M| = p$. Let $\frac{K}{M}$ be a minimal normal subgroup of $\frac{A}{M}$. If $\frac{K}{M}$ is unsolvable, it must be a simple group of order $2^i.3.p$ for some $i, 1 \leq i \leq 7$. But since $p > 19$ there is no K_3-group of such order. So $\frac{K}{M}$ is solvable and therefore elementary abelian, whether it is transitive or intransitive on the partition sets we have $|K| = 20p$. The Sylow p-subgroup of K is normal in K and since it is caracterisitic in K, hence normal in A that contradicting with our assumption $O_p(A) = 1$. Let M be a subgroup of A of oder 10. Since M is intransitive on U, we have that Γ_M is $\frac{A}{M}$-semisymmetric with a bipartition $\{U_M, W_M\}$, $|\frac{A}{M}| = 2^{r+1}.3.p$ and $|U_M| = |W_M| = 2p$. Let $\frac{K}{M}$ be a minimal normal subgroup of $\frac{A}{M}$. If $\frac{K}{M}$ is unsolvable, it must be a simple group of order $2^{i+1}.3.p$ for some $i, 1 \leq i \leq 8$. But there is no K_3-group of such order (for $p > 19$). So $\frac{K}{M}$ is solvable and then elementary abelian, whether it is transitive or intransitive on the partition sets we have $|K| = 20$ or $10p$. The first case could not be possible and in the second case the Sylow p-subgroup of K is normal in K and since it is characteristic in K, hence normal in A that contradicting our assumption $O_p(A) = 1$. Now the lemma is proved. □

Lemma 2.2. *We have either $|O_p(A)| = p$ or $p = 31$ and $\Gamma \cong C(L_2(31); \mathbb{S}_4, \mathbb{S}_4)$.*

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Proof. Let $\{U, W\}$ be a bipartition for Γ. Then $|U| = |W| = 20p$ and $|A| = 2^{r+2}.3.5.p$, for some $r, 0 \leq r \leq 7$. Let N be a minimal normal subgroup of A, then $N \cong T^k$, where T is a simple group. If T is nonabelian, since the powers 3 and 5 in $|A|$ equal 1, we get that $k = 1$ and $N \cong T$. Either $|N|$ divides $|U| = 20p$ or $|U|$ divides $|N|$. In the first case, since $|N|$ is divisible by at least three distinct primes we have $|N| = 2^i.5.p$ and hence N is a simple K_3-group. But the order of such groups is divisible by 3. Therefore $|N|$ is divisible by $20p$. Again since the order of every simple K_3-group is divisible by 3, N must be a simple K_4-group of order $2^i.3.5.p$. So $N \cong L_2(11), L_2(2^4)$ or $L_2(31)$, these groups correspond to $p = 11, p = 17$ and $p = 31$. Since $p > 19$, $N \cong L_2(31)$ and the order of $\frac{A}{N}$ does not divisible by 3. Now Γ is a N-semisymmetric graph. For each $u \in U$ and $v \in W$ by [9], we have $N_u = N_v \cong S_4$ and $\Gamma \cong C(N; N_u, N_v)$. We note that by [24, Theorem 1.3], $C(L_2(31); S_4, S_4)$ is a connected cubic semisymmetric graph.

Now assume that N is solvable and hence elementary abelian. It gives us that Γ_N is a connected cubic $\frac{A}{N}$-semisymmetric graph of order $\frac{40p}{|N|}$. We claim that $|O_p(A)| = p$. Assume not and $O_p(A) = 1$, then $N \cong Z_2, Z_2 \times Z_2$ or Z_5 and Γ_N is $\frac{A}{N}$-semisymmetric graph of order $20p, 10p$ or $8p$ respectively. Take $\{U_N, W_N\}$ to be the bipartition of Γ_N and $\frac{M}{N}$ be a minimal normal subgroup of $\frac{A}{N}$. Assume that $N \cong Z_2$, then $|\frac{M}{N}| = 2^{r+1}.3.5.p$ and $|U_N| = |W_N| = 10p$. If $\frac{M}{N}$ is unsolvable, then it must be a simple K_4-group of order $2^i.3.5.p$ that contradicts our assumption. If $\frac{M}{N}$ is solvable, then $\frac{M}{N} \cong Z_2, Z_5$ or Z_p. If $\frac{M}{N} \cong Z_5$, then$|M| = 10$ that according to lemma 2.1 it could not be possible. If $\frac{M}{N} \cong Z_p$, the Sylow p-subgroup of M is normal and characteristic in M, so it is normal in A, contradicting our assumption. If $\frac{M}{N} \cong Z_2$, then $M \cong Z_2 \times Z_2$ or Z_4, $|\frac{A}{M}| = 2^r.3.5.p$ and $|U_M| = |W_M| = 5p$.Let $\frac{K}{M}$ be a minimal normal subgroup of $\frac{A}{M}$, then it is solvable and isomorphic to Z_5 or Z_p. The first case by lemma 2.1 can not be hold and in the second case we have $|O_p(A)| > 1$, contradiction our assumption. This shows that N is not isomorphic to Z_2 and a similar argument also shows that N is not isomorphic to $Z_2 \times Z_2$. Finally we assume that $N \cong Z_5$, then $|\frac{A}{N}| = 2^{r+2}.3.p$. But there is no such simple K_3-group since $p > 19$. If $\frac{M}{N}$ is solvable, then it is isomorphic to $Z_2, Z_2 \times Z_2$ or Z_p which implies that $|M| = 10, 20$ or $5p$ and each of them is impossible. Now the lemma is proved. □

By lemma above, in the remaining of this paper we assume that $p \neq 31$. Then a Sylow p-subgroup of A is normal in A.

Lemma 2.3. *Let M be the Sylow p-subgroup of A and $\frac{A}{M} \cong G$. Then we have*

(i) *For each vertex u, A_u is isomorphic to a subgroup of G.*
(ii) *$A \cong G \ltimes_\phi M$ for some homomorphism $\phi : G \to Aut(M)$.*

Lemma 2.4. *Let M be the Sylow p-subgroup of A. Then $\frac{A}{M}$ is not isomorphic to A_5.*

Lemma 2.5. *Let M be the Sylow p-subgroup of A. Then $\frac{A}{M}$ is not isomorphic to S_5.*

Now we can prove our main theorem. We note that $\mathbb{F}_{40}$ is the unique cubic symmetric graph of order 40.

The proof of theorem 1.1. : Let Γ be a connected cubic semisymmetric graph of order $40p$. By [7] we assume that $p > 19$. Let $A = \mathrm{Aut}(\Gamma)$ and M be a Sylow p-subgroup of A. We have $|A| = 2^{r+2}.3.5.p$ and by lemma 2.2, either $M \trianglelefteq A$ or $\Gamma \cong C(L_2(31); \mathbb{S}_4, \mathbb{S}_4)$. So we assume that M is normal in A. Let U, W be a bipartition of Γ. Then we have $|U| = |W| = 20p$ and M is on both U and W intransitive. Now Γ_M is a connected cubic $\frac{A}{M}$-semisymmetric graph of order 40. Set $G = \frac{A}{M}$. Since Γ_M is G-semisymmetric, we get by [7] that it is G-edge transitive and hence it is symmetric. Thus $\Gamma_M \cong \mathbb{F}_{40}$ and G is isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{F}_{40})$. By [7] we get that $|\mathrm{Aut}(\mathbb{F}_{40})| = 480$, so we have $|G| = 2^{r+2}.3.5 \leq 480$. This implies that $2^{r+2} \leq 32$ and G is intransitive on U. Therefore $|G| < 480$ and $0 \leq r \leq 2$. Hence $|G| = 60, 120$ or 240 and G is transitive on both U_M and W_M. This gives us that G is a transitive permutation group of degree 20 and of order $60, 120$ or 240. We note that $\mathrm{Aut}(\mathbb{F}_{40})$ has a subgroup $H \cong \mathbb{A}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ of index 2 and G is a subgroup of $\mathrm{Aut}(\mathbb{F}_{40})$. So $G \cap H$ is a subgroup of index at most 2 in G. Assume that $|G| = 60$. Then using Gap we get that $G \cong \mathbb{A}_5$ or $\mathbb{Z}_5 \times \mathbb{A}_4$. The first case due to lemma 2.4 is not true and by the structure of $(\mathbb{F}_{40})$ we get that G is not isomorphic to $\mathbb{Z}_5 \times \mathbb{A}_4$. Now assume that $|G| = 120$. Again using Gap we obtain that $G \cong \mathbb{S}_5, \mathbb{Z}_2 \times \mathbb{A}_5, \mathbb{S}_4 \ltimes \mathbb{Z}_5, \mathbb{Z}_5 \times \mathbb{A}_4$ or $\mathbb{D}_{10} \times \mathbb{A}_4$. By lemma 2.5, G is not isomorphic to S_5. By the structure of $\mathrm{Aut}(\mathbb{F}_{40})$, we get that the only possibility for G is to be isomorphic to $\mathbb{Z}_2 \times \mathbb{A}_5$. If $G \cong \mathbb{Z}_2 \times \mathbb{A}_5$, then G has a normal subgroup $K \cong \mathbb{A}_5$ and 3 does not divide the order of $\frac{G}{K}$. This implies that Γ_M is K-semisymmetric which is impossible. Now assume that $|G| = 240$. Then G is a subgroup of index 2 in $\mathrm{Aut}(\mathbb{F}_{40})$. By this and Gap we get that $G \cong \mathbb{Z}_2 \times \mathbb{S}_5, \mathbb{Z}_4 \times \mathbb{A}_5, \mathbb{Z}_2 \times \mathbb{Z}_2 \times A_5$ or $\mathbb{A}_5 \ltimes \mathbb{Z}_4$. Assume that $G \cong \mathbb{Z}_2 \times \mathbb{S}_5$ and set $T = Z(G)$. Then $T \cong \mathbb{Z}_2$ and $(\Gamma_M)_T$ is $\frac{G}{T}$-semisymmetric. Now let $B = \frac{G}{T}$. Then $B \cong S_5$ and for each edge $\{u, v\}$ in Γ we have $|B_u| = |B_v| = 12$, also each subgroup of B of order 12 is isomorphic to $\mathbb{A}_4$. This gives us that $(B_u, B_v) = (\mathbb{A}_4, \mathbb{A}_4)$ that is impossible. A similar argument shows that G is not isomorphic to $\mathbb{A}_5 \ltimes \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{A}_5$ or $Z_2 \times \mathbb{Z}_2 \times \mathbb{A}_5$. This completes the proof of theorem 1.1. □

3. Conclusions

Corollary 3.1. *We assume p is prime and Γ is a connected cubic graphs of rder$40p$:*

(1) *If Γ is a semisymmetric graph then $p = 3$ and Γ is isomorphic to a semisymmetric cubic graph of order 120 or $p = 31$ and Γ is isomorphic to the coset graph $C(L_2(31) : \mathbb{S}_4, \mathbb{S}_4)$*

(2) *IF $p \neq 3, 31$ then Γ is symmetric*

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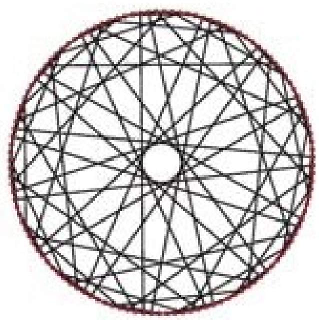
4. **Tables, Figurs and graphs**

Table 1. K_4-Simple groups of order $2^i \times 3 \times 5 \times p$

Group	order
$L_2(11)$	$2^2 \times 3 \times 5 \times 11$
$L_2(2^4)$	$2^4 \times 3 \times 5 \times 17$
$L_2(31)$	$2^5 \times 3 \times 5 \times 31$



S120

Figure 1. Semisymmetric graph diagram $\mathbb{S}_{120}$

References

[1] M. Alaeiyan and M. Ghasemi, Cubic edge-transitive graphs of order $8p^2$, *Bull. Aust. Math.Soc.*, **77** (2008) 315–323.

[2] M. Alaeiyan and B. N. Onagh, Cubic edge-transitive graphs of order $4p^2$, *Acta. Math. Univ. Com. New. Ser.*, **78** (2009) 183–186.

[3] M. Alaeiyan and B. N. Onagh, On semisymmetric cubic graphs of order $10p^3$, *Hace. J. Math. Stat*, **40** (2011) 531–535.

[4] M. Alaeiyan and M. Lashini, Classification of cubic edge-transitive graphs of order $14p^2$, *Hace. J. Math. Stat.*, **41** (2012) 277–282.

[5] P. M. Amoli, M. R. Darafsheh and A. Tehranian, Semi-symmetric cubic graph of order $12p^3$, *Bull. Korean. Math. Soc.*, **59** (2022) 203–212.

[6] Y. Bugeand, Z. Cao and M. Mignoto, On simple K_4-groups, *J. Algebra.*, **241** (2001) 658-668.

[7] M. Conder, M. Malnic, D. Marusic and P. Potocnic, A census of semisymmetric cubic graphs on up to 768 vertices, *J. Algebraic Comb.*, **23** (2006) 255–294.

[8] M. Conder and R. Nedla, A refined classification of symmetric cubic graphs, *J. Algebra.*, **322** (2009) 722–740.

[9] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of finite groups*, Oxford University Press, 1985.

[10] M. Darafsheh and M. Shasavaran, Semisymmetric cubic graphs of order $34p^3$, *Bull. Korean. Math. Soc.*, **57** (2020) 739–750.

[11] M. R. Darafsheh and M. Shahsavaran, On semisymmetric cubic graphs of order $20p^2$, *Discuss. Math. Graph Thory.*, **41** (2021) 873–891.

[12] Y. Feng, M. Ghasemi and W. Changqun, Cubic semisymmtric grphs of order $6p^3$, *Disc. Math.*, **310** (2010) 2345–2355.

[13] J. Folkman, Regular line-symmetric graphs, *J. Comb. Theory.*, **3** (1967) 215–232.

[14] C. Godsil, G. F. Royle, *Algebraic graph theory*, New York, Springer, 2001.

[15] D. M. Goldschmidt, Automorphisms of trivalent graphs, *Ann. Math.*, **111** (1980) 377–406.

[16] H. Han and Z. Lu, Semisymmetric graphs of order $6p^2$ and prime valency, *Sci. China Math.*, **55** (2012) 2579–2592.

[17] M. Herzog, On finite simple groups of order divisible by three primes only, *J. Algebra.*, **120** (1968) 383–388.

[18] X. Hua and Y. Feng, Cubic semisymmetric graphs of order $8p^3$, *Sci. China Math.*, **54** (2011) 1937–1949.

[19] X. Hua, S. Guo and L. Chen, Cubic Semisymmetric Graphs of Order $2qp^2$, *Acta. Math. Appl. Sin. Engl. Ser.*, **35** (2019) 629–637.

[20] J. H. Kwak and R. Nedela, Graphs and their coverings, *Lect. Notes. Ser.*, **17** (2007).

[21] Z. Lu, C. Wang and M. Xu, On semisymmetric cubic graphs of order $6p^2$, *Sci. China Ser. A. Math.*, **47** (2004) 1–17.

[22] A. Malnic, D. Marusic and C. Wang, Cubic Semisymmetric Graphs of Order $2p^3$, *Disc. Math.*, 2000.

[23] A. Malnic, D. Marusic, and C. Wang, Cubic edge-transitive graphs of order $2p^3$, *Disc. Math.*, **274** (2004) 187–198.

[24] C. Parker and P. Rowley, Classical groups in dimension 3 as completions of the Goldschmidt G_3-Amalgam, *J. LMS.*, **62** (2000) 802–812.

[25] D. J. Robinson, *A course in the theory of groups*, Springer-Verlag, New York, 1982.

[26] W. J. Shi, On simple K_4-groups, *Chin. sci. Bull.*, **36** (1991) 1281–1283.

[27] M. Suzuki, *Group theory I*, Springer Berlin, Heidelberg, 1982.

[28] M. Suzuki, *Group theory II*, Springer-Verlag, New York, 1986.

[29] M. Shahsavaran and M. R. Darafsheh, Classifying semisymmetric cubic graphs of order 20p, *Turk. J. Math.*, **43** (2019) 2755–2766.

[30] W. T. Tutte, *Connectivity in graphs*, University of Toronto Press, Toronto, 1966.

[31] S. Zhang and W. J. Shi, Revisiting the number of simple K_4-groups, *arXiv:1307.8079v1 [math.NT]*, (2013).

Mohammad reza Salarian

Faculty of Mathematical Sciences and Computer, Kharazmi of TEHRAN, P.O.Box 1561836314, Tehran, Iran

Email: Salariankhu.ac.ir

Javanshir Rezaei

Faculty of Mathematical Sciences and Computer, Kharazmi of TEHRAN, P.O.Box1561836314, Tehram, Iran

Email:jre927gmail.com

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