http://math-sci.ui.ac.ir

Journal of Mathematics and Society
ISSN (print): 2345-6493, ISSN (on-line): 2345-6507
Vol. 10 No. 1 (2025), pp. 105-124.

http://www.ui.ac.ir

NEW SOLUTIONS TO EINSTEIN'S EQUATIONS TO FIND WALKER MANIFOLDS

SHAHROUD AZAM

Abstract. In this paper, we investigate the Einsteinian manifolds with parallel null distribution. For this, we first obtain the equations that lead to finding the mentioned manifolds. These equations are known as Einstein's equations. Then we reduce these equations by using Lie symmetry method. These equations are known as Einstein's equations. In this method, we first obtain the generators of the symmetry algebra and then calculate the differential invariants for each of the generators and calculate the group invariant solutions of this equation. In addition to this, we also obtain the optimal system of the one-dimensional sub-algebras of these equations. This optimal system helps us to have a classification on group invariant solutions using conjugate mapping.

1. Introduction

Many physical phenomena around us are modeled based on a differential equation. The role of differential equations, especially differential equations with partial derivatives, is very prominent in various sciences, especially in physics and engineering sciences. Obtaining the solutions of such equations has been of interest to mathematicians for a long time, and so far various methods have been presented to solve differential equations. One of the most powerful methods is Lie symmetry method, which was first described by Sophus Lie in the late 19th century [9]. The main and fundamental

Keywords: Solutions of the partial invariants, Walker manifolds, Einstein equation.

Article Type: Research Paper.

Communicated by Mohamad Reza Pouryayevali.

Received: 07-02-2024, Accepted: 06-07-2024, Published Online: 12-04-2025.

Cite this article: Sh. Azam, New solutions to Einstein's equations to find Walker manifolds, *Journal of Mathematics and Society*, **10** no. 1 (2025) 105–124.

http://dx.doi.org/10.22108/msci.2024.140664.1643 .

concept in this method are Lie groups. With this method, the order of ordinary differential equations can be reduced, differential equations with partial derivatives can also be reduced in special cases, the solutions of equations can be classified and etc [4, 12]. In short, it can be said that the symmetric group is a group of transformations that affects the independent and dependent variables of the equation and covert the solutions to solutions. Therefore, by having a symmetric group and a solution to the equation, various solutions of that equation can be obtained. The important point is that in some systems of partial differential equations, all symmetry reductions are not obtained from the classical symmetry method. This caused many mathematicians to search for ways to generalize Lie's method so that with the help of these methods they can obtain more solutions of the equation. Ozyannikov in [13, 15] generalized the concept of group invertible solutions and called this group of solutions partial invertible solutions. The difference between these solutions and group invertible solutions is that their graph is not completely invertible under the group of assumed transformations.

There are more recent generalizations of the concept of group irreducible solutions, which can be referred to the method presented by Blueman and Cole [3], and it is called the non-classical method in group irreducible solutions. In recent years, the solutions of many important and practical equations in physics have been obtained by the partial invariant method, and this method has attracted the attention of researchers in this field [1, 2, 11].

In this article, we are going to get the partial solutions of Einstein's equations and compare them. Einstein's equations are equations whose solutions determine the metric of a special type of manifolds called Walker manifolds. Walker manifolds are manifolds that have a null parallel distribution [5, 17, 18]. These manifolds play a very important role in modeling physical problems. In this, the role of 4-dimensional Walker manifolds is very important. These models express the fact that matter determines the geometry of space-time and opposite to the motion of matter, it is determined by the space meter tensor [16, 19]. Einstein's equation system for 4-dimensional Walker manifolds is:

$$\begin{aligned}
&c_{22}+a_{12}=0, \qquad b_{22}-a_{11}=0, \qquad c_{11}+b_{12}=0,\\
&2c_{23}-ac_{12}-2a_{24}+ba_{22}+2ca_{12}-{c_2}^2-a_2c_1+a_2b_2+a_1c_2=0,\\
&c_{24}-bc_{22}-bc_{22}-cc_{12}x-ac_{11}-b_{23}-a_{14}+ca_{11}-c_1c_2+a_2b_1=0,\\
&2c_{14}-bc_{12}-2b_{13}+2cb_{12}+ab_{11}-{c_1}^2+b_2c_1-b_1c_2+a_1b_1=0,
\end{aligned} \tag{1.1}$$

where a, b, and c are function with respect to (x,t,y,z) and the metric of 4-dimensional Walker manifold is determined by

$$\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & a & c\\ 0 & 1 & c & b \end{pmatrix}. \tag{1.2}$$

http://dx.doi.org/10.22108/msci.2024.140664.1643

Due to the importance expressed about Einstein's equations, many efforts have been made to solve these equations in recent years. In some studies, solutions of this system of equations have been presented in special cases. For example, if we assume that the functions a, b and c depend only on y and z, we can define a hyper-para-Kaehler structure on the manifold which is Ricci flat. Some other special cases have also been investigated, for instance, $a = b = 0$ and $b = c = 0$ [5]. Also, the case where the functions a, b and c depend only on x and t has also been investigated [8]. Also, see [6, 7, 10, 14].

2. Main Results

We prove any two-dimensional sub-algebra of the symmetric algebra of Einstein's equations, with exactly one of the following sub-algebras is conjugated

$$\begin{array}{lll}
\mathcal{V}_1^1 : \langle X_1, X_3 + \alpha X_6 + \beta X_7\rangle, & \mathcal{V}_1^2 : \langle X_1, X_2 + \alpha X_5 + \beta X_7\rangle, & \mathcal{V}_1^3 : \langle X_1, X_5 + \alpha X_7\rangle, \\
\mathcal{V}_1^4 : \langle X_1, X_3 + X_6 + \epsilon X_5 + \alpha X_7\rangle, & \mathcal{V}_1^5 : \langle X_1, X_2 + \alpha X_3 + \beta X_7\rangle, & \mathcal{V}_1^6 : \langle X_1, \alpha X_7 + X_6\rangle, \\
\mathcal{V}_1^7 : \langle X_1, X_7\rangle, & &
\end{array}$$

$$\begin{array}{lll}
\mathcal{V}_2^1 : \langle X_2, \beta X_7 + \alpha X_6 + X_3\rangle, & \mathcal{V}_2^2 : \langle X_2, X_1 + \epsilon X_4 + \beta X_7\rangle, & \mathcal{V}_2^3 : \langle X_2, X_4 + \alpha X_7\rangle, \\
\mathcal{V}_2^4 : \langle X_2, X_3 + \epsilon X_4 + \alpha X_7 + X_6\rangle, & \mathcal{V}_2^5 : \langle X_2, X_1 + X_6 + \alpha X_7\rangle, & \mathcal{V}_2^6 : \langle X_2, \alpha X_7 + X_6\rangle, \\
\mathcal{V}_2^7 : \langle X_2, X_7\rangle, & &
\end{array}$$

$$\begin{array}{lll}
\mathcal{V}_3^1 : \langle X_6, \alpha X_7 +, X_3\rangle, & \mathcal{V}_3^2 : \langle X_6, X_4\rangle, & \mathcal{V}_3^3 : \langle X_6, X_5\rangle, \\
\mathcal{V}_3^4 : \langle X_6, X_1 + \alpha X_7\rangle, & \mathcal{V}_3^5 : \langle X_6, X_2\rangle, & \mathcal{V}_3^6 : \langle X_6, X_7\rangle,
\end{array}$$

$$\mathcal{V}_4^1 : \langle \varepsilon X_1 + X_6, X_2\rangle, \quad \mathcal{V}_4^2 : \langle \varepsilon X_1 + X_6, X_5\rangle, \quad \mathcal{V}_4^3 : \langle \varepsilon X_1 + X_6, X_7\rangle,$$

$$\begin{array}{lll}
\mathcal{V}_5^1 : \langle X_5, X_3 + \alpha X_6 + \beta X_7\rangle, & \mathcal{V}_5^2 : \langle X_5, X_1 + \alpha X_6 + \beta X_7\rangle, & \mathcal{V}_5^3 : \langle X_5, X_6 + \alpha X_7\rangle, \\
\mathcal{V}_5^4 : \langle X_5, X_7\rangle, & &
\end{array}$$

$$\mathcal{V}_6^1 : \langle \varepsilon X_2 + X_5, X_3 + \tfrac{1}{2} X_6 + \alpha X_7\rangle, \quad \mathcal{V}_6^2 : \langle \varepsilon X_2 + X_5, X_1 + \alpha X_7\rangle, \quad \mathcal{V}_6^3 : \langle \varepsilon X_2 + X_5, X_7\rangle,$$

$$\begin{array}{lll}
\mathcal{V}_7^1 : \langle X_4, X_3 + \alpha X_6 + \beta X_7\rangle, & \mathcal{V}_7^2 : \langle X_4, X_2 + \alpha X_3 + \beta X_7\rangle, & \mathcal{V}_7^3 : \langle X_4, X_6 + \alpha X_7\rangle, \\
\mathcal{V}_7^4 : \langle X_4, X_7\rangle, & &
\end{array}$$

$$\mathcal{V}_8^1 : \langle X_4 + \varepsilon X_1, \alpha X_7 + 2X_6 + X_3\rangle, \quad \mathcal{V}_8^2 : \langle \varepsilon X_1 + X_4, X_2 + \alpha X_7\rangle, \quad \mathcal{V}_8^3 : \langle \varepsilon X_1 + X_4, X_7\rangle,$$

$$\begin{array}{lll}
\mathcal{V}_9^1 : \langle X_3, \alpha X_7 + X_2\rangle, & \mathcal{V}_9^2 : \langle X_3, X_5\rangle, & \mathcal{V}_9^3 : \langle X_3, X_1\rangle, \\
\mathcal{V}_9^4 : \langle X_3, X_6 + \alpha X_7\rangle, & \mathcal{V}_9^5 : \langle X_3, X_7\rangle, & \mathcal{V}_9^6 : \langle X_3, X_4\rangle,
\end{array}$$

$$\begin{array}{lll}
\mathcal{V}_{10}^1 : \langle \varepsilon X_2 + X_3, X_1\rangle, & \mathcal{V}_{10}^2 : \langle \varepsilon X_2 + X_3, X_4\rangle, & \mathcal{V}_{10}^3 : \langle \varepsilon X_3 + X_2, X_7\rangle, \\
\mathcal{V}_{10}^4 : \langle X_2, \alpha X_7 + X_3\rangle, & &
\end{array}$$

$$\begin{array}{lll}
\mathcal{V}_{11}^1 : \langle \varepsilon X_4 + X_3, -2X_7 + X_6 + \varepsilon X_5\rangle, & \mathcal{V}_{11}^2 : \langle X_3 + \varepsilon X_4, X_2 + \alpha X_7\rangle, & \mathcal{V}_{11}^3 : \langle \varepsilon X_4 + X_3, X_7\rangle, \\
\mathcal{V}_{11}^4 : \langle \varepsilon X_4 + X_3, X_3 + X_6 + \alpha X_7\rangle, & \mathcal{V}_{11}^5 : \langle \varepsilon X_4 + X_3, \varepsilon X_2 + X_1\rangle, &
\end{array}$$

$$\mathcal{V}_{12}^1 : \langle \varepsilon' X_4 + X_3 + \varepsilon X_2, \varepsilon' X_2 + X_1\rangle, \quad \mathcal{V}_{12}^2 : \langle \varepsilon' X_4 + X_3 + \varepsilon X_2, X_7\rangle,$$

where α and β are constants, $\varepsilon = \pm 1$ and $\varepsilon' = \pm 1$ and $\epsilon = \pm 1$ or 0.

In the following, we calculate the partial invariant solutions of the Einstein equation system.

- For the subalgebra $\langle X_1, X_7 \rangle$ the invariants are $\{t, \frac{b}{a}, \frac{c}{a}\}$, so the dependent variables are

$$a = a(x,t), \qquad c = ag(t), \qquad b = af(t). \tag{2.1}$$

Then the corresponding partial invariants solutions are as

$$1)\begin{cases} a = c_1 \\ b = (c_4 + c_3 t)c_1 \\ c = c_2 c_1 \end{cases} \qquad 2)\begin{cases} a = c_2 + c_1 t \\ b = c_5 + c_3^2 c_1 t \\ c = c_4 c_1 + (c_2 + c_1 t)c_3 \end{cases}.$$

$$3)\begin{cases} a = \dfrac{-\ln(t + c_2)}{c_1} + c_3 \\ b = c_5(c_2 + t) \\ c = c_4 \end{cases} \qquad 4)\begin{cases} a = c_4 + tc_3 + c_1 \ln(c_2 + t) \\ b = \dfrac{c_6^2(t + c_2)}{c_3} \\ c = c_5 + c_6 t \end{cases} \tag{2.2}$$

- For the subalgebra $\langle X_2, X_7 \rangle$ the invariants are $\{x, \frac{b}{a}, \frac{c}{a}\}$, so the dependent variables are

$$c = ag(x), \; b = af(x), \; a = a(x,t).$$

Then the corresponding partial invariants solutions are as

$$a = c_1 x + c_2, \; b = c_1 c_3^2 x + (\frac{c_5}{c_1} - c_3^2 c_2)\ln(c_1 x + c_2) + c_6, \; c = c_3(c_1 x + c_2) + c_1 c_4,$$

.

- For the subalgebra $\langle X_5, X_7 \rangle$ the invariants are $\{t, \frac{c^2 - ba}{b^2}, \frac{-ct + bx}{b}\}$ so the dependent variables are

$$a = -bf(t) + \frac{b(g(t) - x)^2}{t^2}, \; b = b(x,t), \; c = (g(t) - x)\frac{b}{t},$$

Then the corresponding partial invariants solutions are as

$$a = \frac{(c_1 + c_2 t^3)(x - c_3)^2}{t^3} - (A), \; b = \frac{c_1 + c_2 t^3}{t}, \; c = \frac{(c_1 + c_2 t^3)(x - c_3)}{t^2},$$

where

$$A = (c_5 + c_4)t(\ln(t + (\frac{c_1}{c_2})^{(\frac{1}{3})})^2 - \ln(-t(\frac{c_1}{c_2})^{(\frac{1}{3})} + t^2 + (\frac{c_1}{c_2})^{(\frac{2}{3})}) + 2\sqrt{3}\tan^{-1}(\frac{2c_2 t}{\sqrt{3}c_1}(\frac{c_1}{c_2})^{(\frac{2}{3})} - \frac{1}{\sqrt{3}})).$$

- For the subalgebra $\langle X_4, X_7 \rangle$ the invariants are $\{x, \frac{-at+cx}{xa}, \frac{at^2 - 2xtc + bx^2}{x^2 a}\}$so the dependent variables are

$$a = a(x,t), \; b = a(2\frac{t}{x}f(x) + g(x) + \frac{t^2}{x^2}), \; c = a(\frac{t}{x} + f(x))$$

Then the corresponding partial invariants solutions are as

$$a = \frac{c_1 + c_2 x^3}{x}, \; b = \frac{(c_1 + c_2 x^3)(t + c_3)^2}{x^3} + (B), \; c = \frac{(c_1 + c_2 x^3)(t + c_3)}{x^2},$$

http://dx.doi.org/10.22108/msci.2024.140664.1643

where

$$B = (c_5 + c_4)x(\ln(x + (\frac{c_1}{c_2})^{(\frac{1}{3})})^2 - \ln(-x(\frac{c_1}{c_2})^{(\frac{1}{3})} + x^2 + (\frac{c_1}{c_2})^{(\frac{2}{3})}) + 2\sqrt{3}\tan^{-1}(\frac{2c_2x}{\sqrt{3}c_1}(\frac{c_1}{c_2})^{(\frac{2}{3})} - \frac{1}{\sqrt{3}})).$$

- For the subalgebra $\langle X_2, X_6 + X_7\rangle$ the invariants are $\{x, \frac{b}{a^3}, \frac{c}{a^2}\}$, so the dependent variables are

$$c = a^2 g(x),\ b = a^3 f(x),\ a = a(x,t),$$

Then the corresponding partial invariants solutions are as

$$a = \frac{4(t + c_1)}{(c_2x + c_3)^2},\ b = \frac{4c_2^2(t + c_1)^3}{(c_2x + c_3)^4},\ c = \frac{4c_2(t + c_1)^2}{(c_2x + c_3)^3},$$

where c_i are constants.

3. Conclusions

In this article, after stating the definition and the method of obtaining the solutions of differential invariants for an equation, we obtained the two-dimensional sub-algebras of the symmetric algebra of Einstein's equations. Then, using it, we calculated the solutions of the partial invariants of these equations and proved that these solutions are different from the solutions obtained by the Lie symmetry method. In fact, it was proved that these solutions are non-reduceing. Based on this, new and valuable solutions were obtained from one of the famous and widely used equations in physics, Einstein's equations. These equations actually determine the metric of Walker manifolds, which play an essential role in space-time modeling.

References

[1] A. M. Barlukova and A. P. Chupakhin, Partially invariant solutions in gas dynamics and implicit equations, *J. Appl. Mech. Tech. Phys.*, **53** no. 6 (2012) 812–824.

[2] V. B. Bekezhanova and O. N. Goncharova, Application of a partially invariant exact solution of the thermosolutal convection equations for studying the instability of an evaporative flow in a channel heated from above, *Symmetry*, **15** no. 7 (2023) p. 13.

[3] G. W. Bluman and J. D. Cole, The general similarity solution of the heat equation, *J. Math. Mech.*, **18** (1968/69) 1025–1042.

[4] G. W. Bluman and S. Kumei, *Symmetry and differential equations*, Applied mathematical sciences, no. 81, Spriger-Verlag, New York, 1989.

[5] M. Brozos-Vázquez, E. García-Río, P. Gilkey, S. Nikvčević and R. Vázques-Lorenzo, *The geometry of walker manifolds*, A Publication in the Morgan and Claypool Publishers series, 2009.

[6] A. M. Grundland and L. Lalague, Invariant and partially-invariant solutions of the equations describing a non-stationary and isentropic flow for an ideal and compressible fluid in $(3+1)$ dimensions, *J. Phys. A*, **29** no. 8 (1996) 1723–1739.

[7] A. M. Grundland, P. Tempesta and P. Winternitz, Weak transversality and partially invariant solutions, *J. Math. Phys*, **44** no. 6 (2003) 2704–2722.

[8] M. Jafari, On 4-dimensional Einsteinian manifolds with parallel null distribution, *Journal of Mathematics and Society*, **8** no. 3 (2023) 55–79. [In Persian]

[9] S. Lie, On integration of a class of linear partial differential equations by means of definite integrals, translation by N. H. Ibragimov, *Archiv der Mathematik*, **6** no. 3 (1881) 328–368.

[10] L. Martina and P. Winternitz, Partially invariant solutions of nonlinear Klein-Gordon and Laplace equations, *J. Math. Phys.*, **33** no. 8 (1992) 2718–2727.

[11] S. V. Meleshko, A particular class of partially invariant solutions of the navier—stokes equations, *Nonlinear Dynam.*, **36** no. 1 (2004) 47–68.

[12] P. J. Olver, *Applications of Lie groups to differential equations*, Second edition. Graduate Texts in Mathematics, **107**, Springer-Verlag, New York, 1993.

[13] L. V. Ovsiannikov, *Group analysis of differential equations*, Translated from the Russian by Y. Chapovsky. Translation edited by William F. Ames. Academic Press, Inc. New York-London, 1982.

[14] L. V. Ovsiannikov and A. P. Chupakhin, Regular partially invariant submodels of the equations of gas dynamic, *J. Appl. Maths Mechs.*,**60** no. 6 (1996) 969–978.

[15] L. V. Ovsiannikov, *Group properties of differential equations*, Novosibirsk, Moscow, 1962.

[16] A. A. Shaikh, Y. H. Kim and S. K. Hui, On Lorentzian quasi-Einstein manifolds, *J. Korean Math. Soc.*, **48** no. 4 (2011) 669–689.

[17] A. G. Walker, Canonical form for a Riemannian space with a parallel field of null planes, *Quart. J. Math. Oxford*, **2** no. 1 (1950) 69–79.

[18] A. G. Walker, On parallel fields of partially null vector spaces, *Quart. J. Math, Oxford*, **20** (1949) 135–145.

[19] A. Zaeim, Y. AryaNejad and M. Gheitasi, On some conformally Einstein manifolds of dimension four, *Journal of Mathematics and Society*, **7** no. 2 (2022) 19–36. [In Persian]

Shahroud Azami

Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University Qazvin, Iran

Email: azami@sci.ikiu.ac.ir

 http://dx.doi.org/10.22108/msci.2024.140664.1643