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GRAPH THEORY; HISTORY, APPLICATIONS AND VISION

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ABSTRACT. Graph theory is a leading theory in mathematics, which is used in many sciences. In this article, by stating the history of this, its expansion and development, we have mentioned the course of famous problems in graph theory. Also, a practical application of this theory is stated.

1. **Introduction**

Perhaps when Leonard Euler, (1707-1783) was thinking about the Konigsberg Bridge Problem for the first time, he did not imagine that later this problem would lead to the creation of a very important, extensive and practical branch of mathematics. Graph theory, has been growing and expanding as an active and dynamic branch. In addition to the growth of this theory as a science, its many applications in other sciences have caused this branch of mathematics to be highly regarded by other scientists in addition to mathematicians. In fact, graphs are efficient mathematical models for analyzing real world problems. Many different problems of real-world are directly related to graph theory, and this theory has come to the aid of other sciences as a powerful tool to solve various problems. From solving the sudoku square to the complex issues of medical science, transportation, network, genetics, etc., all of them can be solved with graph theory or help a lot to get closer to their solution. In this article, we will have an overview of the background of this branch of mathematics, its evolution and applications,

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and finally, we will examine the future prospects for this theory. References [3, 5, 7, 8, 9, 11, 12, 16, 19] have been used to express the basic concepts of graph theory.

2. The Beginning and Expansion of Graph Theory

The birth of graph theory goes back directly to the 18th century and 1736 [10]. At that time, Leonard Euler, a Swiss scientist, was known as a well-known scientist in the world of mathematics. The beautiful city of Konigsberg, (now Kaliningrad), is one of the cities of Russia. Pregel river passes through this city. This river divided the city into four areas, which were connected by seven bridges. The question that arose in the minds of the people of this city while walking on these bridges was whether it is possible to cross these bridges in a way that not only passes all the bridges, but also passes each bridge exactly once? What the people of the city used to do was to try this task throughout the day by trial and error and they never succeeded in doing this task. But Euler's approach as a mathematician was different towards this problem and he tried to look at the subject with a mathematical point of view. He realized that this topic could belong to a new area of mathematics. In Euler's initial solution for this problem, there is no reference to the concepts known today such as graph, vertex and edge, although these concepts are actually used. Euler considered the land points with the names A, B, C and D and called the seven bridges a, b, c, d, e, f and g. In Figure 1, the image created by Euler can be seen.

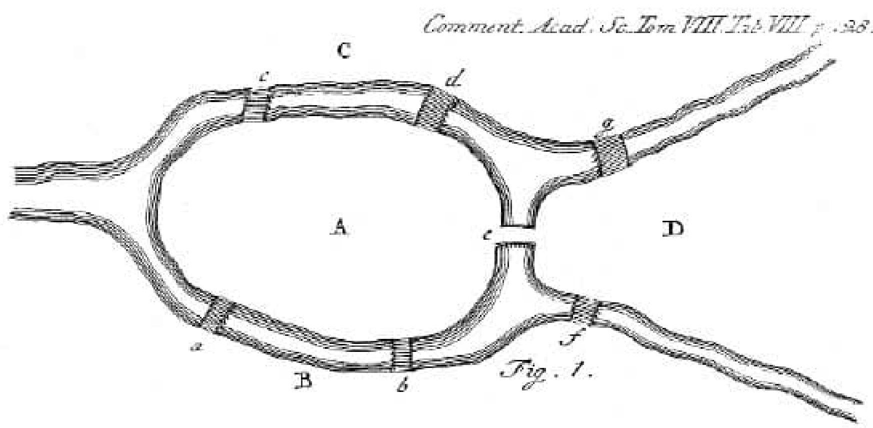

FIGURE 1. The image created by Euler regarding the Konigsberg bridges problem

Euler has presented similar results for a number of other bridges and regions. These results, together with the Konigsberg bridge problem, have been published in his article [10].

If we look at the Konigsberg bridges problem with today's graph theory literature, we will have a graph with four vertices and seven edges, which represent the number of regions and the number of bridges, respectively.

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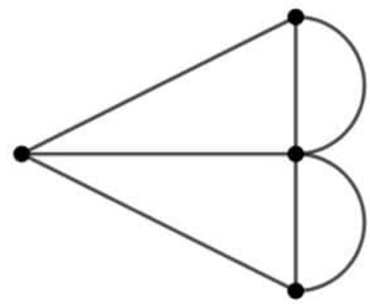

Figure 2. Graph of Konigsberg bridges problem

Based on Euler's argument in solving the bridge problem, new definitions were created in graph theory.

After Euler's initial actions, there was no special development in graph theory for a century, and gradually in the 19th century, this issue attracted the attention of more mathematicians. The term graph was first used in 1878 by British mathematician James Sylvester. In the 19th century, the famous British mathematician, Arthor Kayley, conducted studies in the field of graph theory, which is still recognized as part of the most important results in graph theory. Kayley's work was actually the foundation of graph counting theory and he expressed a complete formula for counting the number of trees on a certain number of vertices, which was named Kayley's formula [12]. In the 20th century, graph theory was recognized as one of the most important branches of mathematics, which created the most applications in other sciences.

3. State Some Important and Historical Problems in Graph Theory

Throughout the history of graph theory, various and interesting problems have arisen. Counting spanning trees, four-color conjecture, Kuratowski's Theorem, traveling salesman problem and Turán's Theorem are examples of these topics. In this section, we will try to briefly address each of the above famous problems.

3.1. **Four-color conjecture.** One of the most important topics in graph theory in the 19th century is the four-color conjecture. This topic examines the coloring of points on a page. We consider a geography map that includes different countries and regions. If we consider each country as a point (vertex) and connect the countries that have a common border (edge), then we will have a graph. Is it enough to have four colors to color a map with the condition that no two borders have the same color? This conjecture was first proposed in 1852 by Francis Gutherie, a mathematician from South Africa, and later it was carefully examined by many mathematicians. Francis was the first to realize that a map can be colored with only four colors. Of course, he did not know the proof of this , and through his brother, he asked De Morgan, who was his professor at University College London, for the solution [15]. De Morgan, discussed the matter with Sir William Rowan Hamilton, who immediately replied that he could not think about it any time soon. Little by little, with the efforts of De Morgan, many mathematicians started to study and investigate this issue. One of the most famous of them was Kayley. He even wrote an article about coloring maps in 1879 [15].

In 1879, one of kayley's students claimed to have a proof of this conjecture, which was published in cooperation with kayley. However, it was violated in 1890 by John Heywood. Haywood showed that this proof is true for 5 colors and not for 4 colors. Efforts to prove the four color conjecture continued and in 1976 the proof of this conjecture was presented [15]. This proof was actually presented by Kenneth Appel and Wolfgang Hicken with the progress of science and the presence of computers in scientific subjects [1, 14]. This was actually the beginning of the presence of computers in mathematical proofs, although so far no purely mathematical proof has been obtained for this problem.

Theorem 3.1 (four colors)**.** *Every planar graph is four colorable.*

3.2. **Kuratowski's Theorem.** At the beginning of the 18th century, the great mathematician Leonard Euler again laid the foundation for the emergence of another important concept in graph theory, planar graphs. Euler obtained an important formula about the relationship between the number of vertices, the number of edges and the number of faces in a polyhedron. After that time and with a long break, many results have been obtained regarding planar graphs. The first of them was when the Polish mathematician, Kuratowski, determined an important criterion for the flatness of a graph. A graph is called planar if we can draw it on a surface in such a way that the edges do not intersect each other except at the vertices. In fact, Euler's formula was the basis of graph topological theory. In 1930, Kuratowski stated and proved an important theorem about planar graphs, which was later known by his name [14, 17].

Theorem 3.2 (Kuratowski)**.** *Every finite graph is planar, if and only if, it does not contain a subgraph isomorphic with a subdivided graph of K_5 or $K_{3,3}$.*

3.3. **Traveling salesman.** Consider a set of cities (regions) and the distances between them. A traveling salesman wants to pass through all cities exactly once and return to the starting point. What is the shortest route that he can take? Note that the problem is not only passing through all the cities and returning to the starting point, but it is also important that this journey has the shortest route. The special case of this problem, in which it was only to find such a path, was proposed by Hamilton. Therefore, there is a difference between finding the Hamiltonian cycle in the graph and the traveling salesman problem. If the graph is complete (there is a path between all the cities), then since there is a Hamiltonian cycle, it will only be a matter of finding a cycle with the shortest

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path. This problem has been studied as a multifaceted issue in computer science, graph theory and optimization for several decades and solutions have been proposed for it. The easiest solution is to try all the ways! But this solution is time-consuming and costly. One of the better solutions is the Monte Carlo algorithm. Although the first academic solution for this problem was expressed in the 1930s, but in the 1950s and 1960s and with the appearanceof computers, this problem was pursued as a famous problem. A lot of work has been done on this problem in optimization theory.

3.4. **Turán's graph Theorem.** Undoubtedly, one of the most important theorems of graph theory is Turán's theorem, which was presented in 1941 by the Hungarian mathematician Paul Turń [4]. This Theorem can be considered as the beginning of Frinal graph theory. First, we consider the definition of cluster and then state Turán's Theorem.

Let G be a graph. A complete subgraph of G is called a cluster. A p-cluster in G is a complete p-vertex subgraph of G denoted by K_p.

Theorem 3.3 (Turń's Theorem)**.** *If $G = (V, E)$ is a graph that contains no copy of K_p and $|V| = n$. Then*

$$|E| \leq \frac{n^2}{2}\left(1 - \frac{1}{p-1}\right)$$

4. An Application of Graph Theory

Graph theory has many applications in different sciences. For example, there are many applications of graph theory in computer science, communication science, networks, physics, chemistry, and nanoscience, which in addition to helping these sciences, has also caused the growth and development of this theory. In this section, we discuss one of the applications of this theory.

Football, as one of the most popular and famous sports in the world, has always attracted the attention of fans. The applications of graph theory in football can be classified in the following cases:

Network analysis of intra-team performance, training pattern modeling, tactical decisions, analysis of competitors' situation, prediction of player performance, team behavior, etc.

Each of the above cases can be analyzed by graphical modeling. If we consider a directed graph and consider each player as a graph vertex and their movement direction as an edge, and even give weight to the edges and consider a measure for each player that shows its performance, We can use this model to optimize training and team performance. Also, one of the attractive applications of graph theory can be found in the ranking of players in various sports fields. Every year, world sports federations publish the rankings of athletes or sports teams according to their performance. This classification cannot be based on the number of wins and losses of athletes and sports teams, because in this case, the rank of each athlete based on his merit and ability is not determined correctly. For example, let's say two players A and B both have two wins, but player A won his two wins against two strong players , but player B won against two weak players . Therefore, the two wins of player A are more valuable than the two wins of player B. Also, the direct performance of players against each other can also be considered as a criterion for ranking (if those players have competed against each other). On the other hand, considering all these situations, two players may have equal points, which should be thought about. All these cases will lead to the necessity of using a mathematical model to determine the correct ranking of athletes. In designing this model, graph theory is used.

5. Vision

Today, graph theory has become a complex world of interdisciplinary concepts and principles that are effectively used in solving problems and making decisions in various areas of human life. This theory is used for cases such as social networks, resource allocation, routing problems, optimization, transportation and computer science. Considering the practical nature of graph theory and the increasing development of science in the world, the increasing growth of artificial intelligence, data science, medicine, chemistry, etc. and the role of graph theory in these fields, it can be predicted that this the theory of the future will play a much stronger role in the development of many sciences. Let's go back to the last century. Hilbert's 1900 speech to the International Congress of Mathematicians in Paris is perhaps the most influential speech ever given by a mathematician on mathematics. In that speech, Hilbert stated 23 main mathematical problems that are to be studied in the next century [6, 13]. These issues deeply influenced mathematics in the last century. Perhaps, if this speech was given in the current century, one of its important topics would be graph theory and its perspective. One of the greatest strengths of graph theory is its abundance of "beautiful problems" and "waiting to be solved". New guesses and claims regarding open and new issues are still being created in this theory. Many problems in graph theory do not require complex tools and premises and rely more on ingenuity and creative thinking. Although the role of mastery in various mathematical concepts such as combinations, probability theory and many other subjects in the progress of this theory is undeniable. Undoubtedly, the application of graph theory in other sciences and the use of this powerful tool in solving various problems, especially in computer science, will be the most important reason for the existence of a brilliant perspective for this theory in the future. However, one should not neglect the inherent beauty of this theory and the wonderful world of graphs as a reason for its development.

References

[1] K. Appel and W. Haken, Every planar map is four colorable, *Illinois Journal of Mathematics*, **21** (1977) 439–597.

[2] M. Behzad and G. Chartrand, *Introduction to the theory of graphs*, Allyn and Bacon: Boston, 1971.

http://dx.doi.org/10.22108/msci.2024.139804.1623

[3] N. Biggs, *Algebraic Graph Theory*, Second edition, Cambridge University Press, Cambridge, 1993.

[4] B. Bollobás, *Extremal Graph Theory*, Academic Press, 1978.

[5] J. A. Bondy and U. S. R. Murty, *Graph Theory, Graduate: Texts in Mathematics*, Springer, New York, 2008.

[6] F. E. Browder (editor), *Mathematical Developments Arising from Hilbert Problems, Proceedings of Symposia in Pure Mathematics*, **28** (Part 1), American Mathematical Society, Providence, 1976.

[7] A. Cayley and M. Philos, 1874, 47, 444–446, as quoted in N. L. Biggs, E. K. Lloyd and R. J. Wilson, *Graph Theory 1736–1936*, Clarendon Press, Oxford, 1976; Oxford, University Press, 1986.

[8] G. Chartrand, L. Lesniak and P. Zhang, *Graphs & digraphs*, (5th ed.), CRC Press, 2012.

[9] R. Diestel, *Graph theory* (5th ed.), Springer, 2017.

[10] L. Euler, Solutio problematis ad geometriam situs pertinentis, *Comment. Acad. Sci. U Petrop*, **8** (1736) 28–40.

[11] C. Godsil and G. Royle, *Algebraic Graph Theory, Graduate Texts in Mathematics*, Springer–Verlag, New York, 2001.

[12] F. Harary, *Graph theory*, Addison-Wesley, 1969.

[13] D. Hilbert; Mathematical problems, *Bulletin of the American Mathematical Society*, **8** (1902) 437–479.

[14] K. Kuratowski, Sur le probléme des courbes gauches en topologie, *Fund Math*, **15** (1930) 271–283.

[15] j. O'Connor and E. F. Robertson, The Four Colour Theorem, 1996. Web. 2 Apr. 2015. http://www-history.mcs.st-and.ac.uk/HistTopics/Thefour

[16] A. J. Schwenk and R. J. Wilson, On the eigenvalues of a graph, *Selected Topics in Graph Theory*, (1978) 307–336.

[17] C. Thomassen, Kuratowski's theorem, *J. Graph Theory*, **5** (1981) 225–241.

[18] G. Wanner, H. Gerhard and E. Hairer, *Analysis by its history*, (1st ed.), Springer Publishing, 2005.

[19] D. West, *Introduction to graph theory*, Prentice Hall, 2001.

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