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TWO PROOFS FROM ĀZARKHOR ASHTĀZ GOSHNASP

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Abstract. One of the Iranian scientists of the fourth and fifth centuries AH is Āzarkhor Ashtāz Goshnasp. His name is only mentioned in the works of the prominent Iranian scientist, Abū Rayhān al-Bīrūnī. In this paper, we will explain Āzarkhor's two proofs on the first theorem of the book "al-Estekhrāj al-Awtār" that given by Abū Rayhān. This theorem is called "Theorem of the broken chord" and these two proofs are the only mathematical legacy left by Āzarkhor.

1. Introduction

One of the Iranian engineers and mathematicians of the second half of the 4th century and the first third of the 5th century AH (10th and 11th centuries AD) is Āzarkhor Ashtāz Goshnasp. There is not much information about his life and his name is only mentioned in the works of famous Iranian scientist, Ūstād Abū Rayhān al-Bīrūnī [5]. In fact, Abū Rayhān's works are the only source about him. His name in Abū Rayhān's book, "al-Āthār al-Bāqiyah 'An al-Qurūn al-Khāliyah" (The Remaining Signs of Past Centuries) or in short "al-Āthār al-Bāqiyah", is mentioned in three places as "Abul-Hasan Āzarkhor ibn Yazdān-Khasis al-Muhandis" (where "Yazdān-Khasis" can be considered as a misspelling of "Yazdān-Djoshnasb"), "Abul-Hasan Āzarkhor al-Muhandis" and "Āzarkhor al-Muhandis" [2]. In these positions, Abū Rayhān has stated what he heard from Āzarkhor himself, respectively about

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"Epahomene", "the opinion of Iranians about the origin of the world and the creation of mankind" and "Sraosha day". In fact, al-Āthār al-Bāqiyah is a book on chronology, mixed with mathematics, astronomy and history, in which Abū Rayhān, in addition to the comparative study of the calendars of different nations, also explores their customs and religions. As can be seen, Āzarkhor was a contemporary of Abū Rayhān and they have met each other at least once. Also, in Abū Rayhān's book, the name "Āzarkhor" appears as "Āzarkhor ibn Ashtāz Djoshnas" in two positions [3]. According to the existence of similar names in the book "Iranian Name Book" [1], it is concluded that the correct form of his name is "Āzarkhor Ashtāz Goshnasp" [6].

The legacy of mathematics left by Āzarkhor can be found in the book "al-Estekhrāj al-Awtār". Abū Rayhān have discussed four geometrical theorems in this book and presented various proofs from himself and other mathematicians for each of them. Then, he have solved 30 problems in geometry, algebra and astronomy using these proofs and arguments. The main purpose of the book "al-Estekhrāj al-Awtār" is to compute the length of the chords of a circle in terms of its diameter [3]. In fact, Abū Rayhān has written two books on the derivation of the chords in the scientific language of those days, namely Arabic. All the contents of these two books have been compiled in a new form and along with the current mathematical terms by Abolqāsem Qorbānī, a prominent researcher of the history of Iranian mathematics, in Persian language in a book titled "Tahrīr al-Estekhrāj al-Awtār" [3]. To prove the first theorem of this book, Abū Rayhān has given 22 proofs, two of which are from Āzarkhor. Āzarkhor's first proof is similar to one of Archimedes's proofs in his book "The Circles" with a slight difference. Abū Rayhān has chosen the second proof of Āzarkhor with a small change to be mentioned in the book "al-Qānūn al-Mas'ūdī" from among the proofs that he and other scientists have given for the theorem [4]. In this paper, we explain Āzarkhor's two proofs for the first theorem of al-Estekhrāj al-Awtār. As it is stated, these two proofs are the only mathematical legacy left by Āzarkhor, which has reached us thanks to Abū Rayhān.

2. Main Results

In this section, we will state the first theorem of the book "al-Estekhrāj al-Awtār" and Āzarkhor's two proofs for this theorem. This theorem is known as "Theorem of the broken chord". To see the theorem and Āzarkhor's proofs, refer to the manuscript of the Leiden University Library. See Figures 1, 2 and 3. Leiden University is a university in the Netherlands whose library contains important resources in oriental studies.

Theorem 2.1. *Let the arc ABC in a circle and the broken chord ABC such that $AB > BC$. If we draw the perpendicular DE from point D, which is the midpoint of the arc ABC, to the chord AB then $AE = BC + BE$, that is, E bisects the broken chord ABC (Figure 4).*

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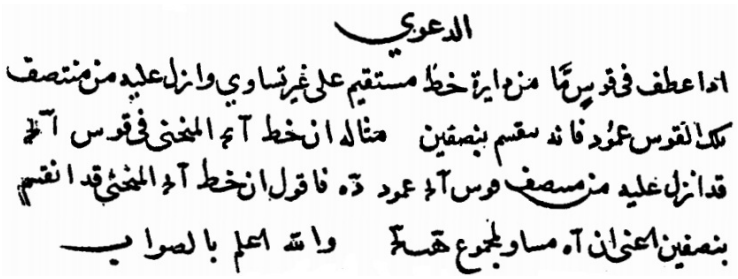

FIGURE 1. The first theorem in the Leiden University Library manuscript.

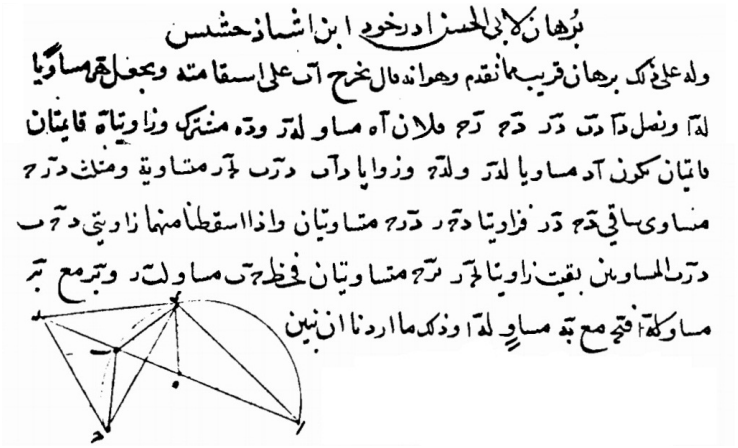

FIGURE 2. Āzarkhor's first proof in the Leiden University Library manuscript.

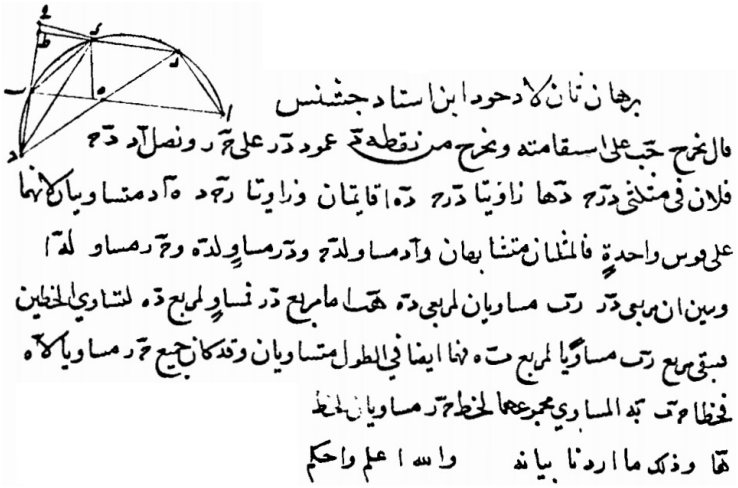

FIGURE 3. Āzarkhor's second proof in the Leiden University Library manuscript.

3. Summary of Proofs

In this section, we explain Āzarkhor's two proofs.

The first proof. We extend the chord AB to the point Z so that $AE = EZ$ and draw the line segments AD, BD, CD, CZ and DZ (Figure 5). Since DE is the perpendicular bisector of AZ, $AD = DZ$ and $\angle DAB = \angle DZB$. It is clear that $\angle DAB = \angle DCB = \overset{\frown}{BD}/2$. On the other hand, D is in the middle point of the arc ABC and $AD = CD$. Therefore, CDZ is an isosceles triangle and $\angle DZC = \angle DCZ$. Now,

$$\angle BZC = \angle DZC - \angle DZB = \angle DCZ - \angle DCB = \angle BCZ$$

and BCZ is also an isosceles triangle. So, $BC = BZ$ and

$$BC + BE = BZ + BE = ZE = AE.$$

□

The second proof. We extend the line segment BC and draw a perpendicular line DZ from D to the line BC. Moreover, we draw the line segments AD, BD and CD (Figure 6). Since D is in the middle point of the arc ABC, $AD = CD$. On the other hand, $\angle DAB = \angle BCD = \overset{\frown}{BD}/2$. Therefore, two right triangles AED and CDZ are congruent by the ASA and $DE = DZ$ and $AE = CZ$. It follows that two right triangles BDE and BDZ, which have a common chord, are congruent. So, $BE = BZ$ and

$$AE = CZ = CB + BZ = CB + BE.$$

□

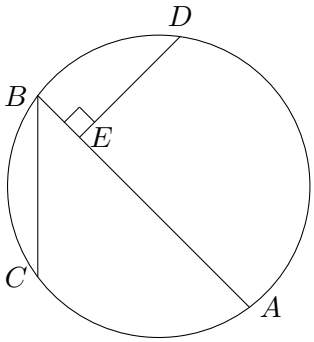

FIGURE 4. The broken line enclosed in a circle

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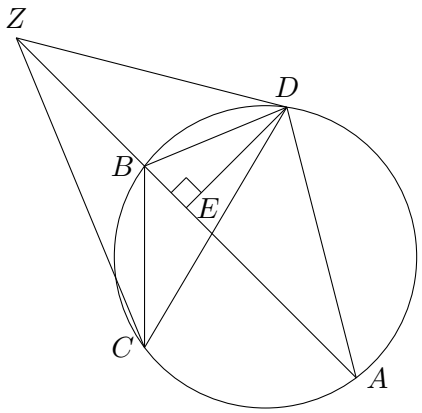

FIGURE 5. Āzarkhor's first proof

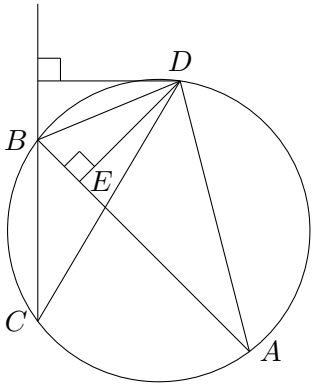

FIGURE 6. Āzarkhor's second proof

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