

ONE ITERATION METHOD IN NUMERICAL SOLUTION OF ENDEMIC MODELS WITH ARBITRARILY DISTRIBUTED PERIODS OF INFECTION FOR LONG TIMES

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ABSTRACT. In this paper, we consider endemic models with arbitrary distributed periods of infection. Some samples of such diseases are: HIV/AIDS, rubella, influenza, and so on. Since beginning of 20th century, there are many authors that investigate about mathematical modelling, existence of solutions and stability of models. Many of these models are in the form of differential equations, integral equations, algebraic equation or hybrid of these. In this work we convert the existence model to a system of integral equations with a nonlinear algebraic equation. We solve numerically this model by an iterative process. Organization of the given algorithm is such that the problem is solved by a numerical iteration on union of short time intervals, and the process forward interval by interval. Convergence of the method is given expansively. In the numerical results section, according to the structure of the problem and by using Laplace transform, we sketch a spectrum of sample problems that have analytical solutions. Finally, we illustrate accuracy and applicability of the method by two benchmark sample problems.

1. Introduction

In this work we consider the spread of an infectious disease in a population with the size $N(t)$ at time t . The population is divided in to suseptiable and infected individuals. $S(t)$ denotes the number

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of susceptible individuals at the time t and $I(t) = \sum_{j=1}^n I_j(t)$ denotes the infected individuals, which are further divided into n stages of infection. Hence the population size is

$$(1.1) \quad N(t) = S(t) + \sum_{j=1}^n I_j(t).$$

The mathematical model of problem is formulated as follows [9]

$$(1.2) \quad \begin{aligned} \dot{S}(t) &= \Lambda - \mu S(t) - B_0(t) + B_n(t), \quad S(0) = \check{S}, \\ B_0(t) &= f(S(t), I_1(t), \dots, I_n(t)), \\ I_j(t) &= \int_0^t B_{j-1}(t-a) F_j(a) P_j(a) e^{-\mu a} da + e^{-\mu t} \check{I}_j(t), \\ B_j(t) &= \int_0^t B_{j-1}(t-a) F_j(a) P'_j(a) e^{-\mu a} da + e^{-\mu t} \check{B}_j(t), \end{aligned}$$

where $j = 1 \dots, n$, F_j, P_j, f are known functions, $\lambda, \mu, \check{S}$ are known constants, and we will describe these quantities in the rest of the paper. The known forcing functions $\check{I}_j(t), \check{B}_j(t)$ are as follows

$$\begin{aligned} \check{I}_j(t) &= \int_t^\infty \frac{\check{u}_j(a-t)}{F_j(a-t)P_j(a-t)} F_j(a) P_j(a) da, \\ \check{B}_j(t) &= \int_0^t \frac{\check{u}_j(a-t)}{F_j(a-t)P_j(a-t)} F_j(a) P'_j(a) da. \end{aligned}$$

where $\check{u}_j$ denotes the stage age density of individuals that were in the j th stage at time $t = 0$, and are known. Other functions, $S(t), B_0(t), B_1(t), \dots, B_n(t), I_1(t), \dots, I_n(t)$ are unknown functions of the model. $\check{S}$ is the initial susceptible individuals, $\lambda > 0$ is the influx rate of new individuals into the epidemiologically relevant part of the population. If the whole population is epidemiologically relevant, $\lambda > 0$ is the population birth rate. $\mu > 0$ is the per capita mortality rate not due to disease-related causes. The function $P_j : [0, \infty) \rightarrow [0, 1]$ describes the duration of the j th stage and the function $F_j : [0, \infty) \rightarrow [0, 1]$ describes the disease-related mortality in the j th stage. P_j and F_j are nonincreasing, nonnegative functions on $[0, \infty)$, and have the following properties

$$\begin{aligned} F_j(0) &= 1 = P_j(0), \\ F_j(a)P_j(a) &> 0, \quad 0 \leq a \leq a_j, \\ F_j(a)P_j(a) &= 0, \quad a_j < a, \text{ when } a_j < \infty, \end{aligned}$$

where a_j denotes the maximum sojourn in the j th stage. The average duration of the j th stage is given by

$$D_j = \int_0^\infty P_j(a) da,$$

where we assume $D_j < \infty$ for all stages j except possibly for the last stage, $j = n$. Feng and Thieme (2000) [9] show that if f satisfies the following hypothesis H

$H : f(S, I_1, \dots, I_n)$ is defined for all $S \geq 0$ and all nonnegative I_j , and is continuous and nonnegative in these variable. Furthermore

$$f(S, 0) = 0, \quad f(0, I) = 0, \quad S \geq 0, I \in [0, +\infty)^n.$$

then the system (1.2) has a unique solution. Hence we Assume that this hypothesis is true. For more details of formulation of the model see [9, 25]. A very general mathematical framework for the analysis of infectious diseases are obtained by Kermack and Mc Kendrick [16, 17, 18]. Stech and Williams (1981) show a remarkable global stability result for the endemic equilibrium in a model with an arbitrarily distributed immunity period [23], their result was recently extended to diseases that cause fatalities [24]. Lin and van den Driessche (1992) prove threshold results for models with an arbitrarily distributed immunity period and a nonlinear incidence [19]. Castillo-Chavez et al. (1989) study the global stability of the disease-free and the local stability of the endemic equilibrium for an AIDS model with arbitrarily distributed infectivity period and a general contact function [8]. The local stability of such models is given by Brauer [4, 5, 6]. Hethcote and Tudor (1980) first consider an SIR model with an arbitrarily distributed infectious period for which they establish stability results for the disease-free and the endemic equilibrium [10]. Babayar-Razlighi have solved this system by the extrapolation method [3]. Asadi-Mehregan et al. (2023) studies the numerical solution of a mathematical model of the Covid-19 pandemic utilizing a meshless local discrete Galerkin method [1]. Asadi-Mehregan et al. (2022) studies on the approximate solution of dynamic systems derived from the HIV infection of CD_4^+ T cells using the LRBF-collocation scheme [2]. In modeling a disease like HIV/AIDS one may like to further divide the infectious stage according to disease progression [11, 21]. A convenient concept to model arbitrary distributions of stage durations is stage (or class) age [12, 13]. Stage age, here usually denoted by a , is the time that has elapsed since entering the stage.

2. Main Results

Integration of the only differential equation in (1.2) on $0 \leq a \leq t$ implies that

$$(2.1) \quad S(t) = \check{S} + \Lambda t + \int_0^t (-\mu S(a) - B_0(a) + B_n(a)) da.$$

Now we define the column vector $X(t)$ as follow

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n+1}(t) \\ x_{n+2}(t) \\ x_{n+3}(t) \\ \vdots \\ x_{2n+2}(t) \end{bmatrix} := \begin{bmatrix} B_0(t) \\ B_1(t) \\ \vdots \\ B_n(t) \\ S(t) \\ I_1(t) \\ \vdots \\ I_n(t) \end{bmatrix}.$$

The operator $T : \mathbb{R}^{2n+2} \rightarrow \mathbb{R}^{2n+2}$ is defined by

$$T(X(t)) := \begin{bmatrix} f(S(t), I_1(t), \dots, I_n(t)) \\ \int_0^t B_0(t-a)F_1(a)P_1'(a)e^{-\mu a}da \\ \vdots \\ \int_0^t B_{n-1}(t-a)F_n(a)P_n'(a)e^{-\mu a}da \\ \int_0^t (-\mu S(a) - B_0(a) + B_n(a))da \\ \int_0^t B_0(t-a)F_1(a)P_1(a)e^{-\mu a}da \\ \vdots \\ \int_0^t B_{n-1}(t-a)F_n(a)P_n(a)e^{-\mu a}da \end{bmatrix} =: \begin{bmatrix} T_1(S(t), I_1(t), \dots, I_n(t)) \\ T_2(B_0(t)) \\ \vdots \\ T_{n+1}(B_{n-1}(t)) \\ T_{n+2}(B_0(t), B_n(t), S(t)) \\ T_{n+3}(B_0(t)) \\ \vdots \\ T_{2n+2}(B_{n-1}(t)) \end{bmatrix}.$$

And finally the known right hand side vector is defined by

$$r(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_{n+1}(t) \\ r_{n+2}(t) \\ r_{n+3}(t) \\ \vdots \\ r_{2n+2}(t) \end{bmatrix} := \begin{bmatrix} 0 \\ e^{-\mu t} \check{B}_1(t) \\ \vdots \\ e^{-\mu t} \check{B}_n(t) \\ \check{S} + \Lambda t \\ e^{-\mu t} \check{I}_1(t) \\ \vdots \\ e^{-\mu t} \check{I}_n(t) \end{bmatrix}.$$

Thus the system (1.2) is equivalent with the following system

$$(2.2) \quad (I - T)(X(t)) = r(t).$$

where I is the identity operator. The main functions of the model are $S(t), I_1(t), \dots, I_n(t)$ and the functions $B_0(t), B_1(t), \dots, B_n(t)$ are the auxiliary functions for obtaining the main functions.

2.1. Algorithm of the method. First of all we define a procedure for approximation of $\int_a^b g(x)dx$ by the Gaussian integration technique. We do this aim in the following subsection

2.1.1. *Algorithm 1 (approximation of $\int_a^b g(x)dx$).* Step 1. Input $g(x), a, b$;

Step 2. Define a local environment with the local variables X, W, dat ;

Step 3. Set $X = \{x_1, \dots, x_{N_G}\}, W = \{w_1, \dots, w_{N_G}\}$, where N_G is the number of Gaussian nodal points, furthermore $x_1, \dots, x_{N_G}$ and $w_1, \dots, w_{N_G}$ are the Gaussian nodal points and Gaussian coefficients on $[a, b]$ respectively;

Step4. Return $dat = \sum_{i=1}^{N_G} w_i g(x_i)$ as the out put of procedure;

We denote out-put of the algorithm 1 by $I_G(g(x), a, b)$. For more details about the Gaussian integration see [7, 14, 15, 22]. Now we can give the main algorithm

2.1.2. *Main Algorithm.* The following algorithm solves system (2.2) on $t \in [0, lN_I] = \sum_{i=1}^{N_I} J_i$, where l is the lenght of short partial intervals, $J_i[l(i-1), li], i = 1 \dots, N_I$ and N_I is the number of intervals.

step 1. Input the positive integers n, N_I, N_G, N_{It} , the reals $\mu, \lambda, \check{S}, l$, the right hand side vector $r(t)$ and the functions $f(S, I_1, \dots, I_n), F_j(a), P_j(a), j = 1 \dots n$. Set $a = 0, b = l$,

$$U^{(0)}(t) = \begin{bmatrix} \begin{cases} \check{S} & t \leq b, \\ 0 & \text{Otherwise.} \end{cases} \\ \begin{cases} r_{n+3}(0) & t \leq b, \\ 0 & \text{Otherwise.} \end{cases} \\ \vdots \\ \begin{cases} r_{2n+2}(0) & t \leq b, \\ 0 & \text{Otherwise.} \end{cases} \end{bmatrix}, \tilde{B}_j(t) = \begin{cases} r_{j+1}(0) & t \leq b, \\ 0 & \text{Otherwise.} \end{cases}, j = 1, \dots, n.$$

step 2. In accordance with the Algorithm 2.1.1, define the procedure $I_G(g(x), a, b)$;

step 3. Set $i_1 = 1$;

step 3.1 If $i_1 > N_I$, then go to step 4.;

step 3.2 Set $i_2 = 1$;

step 3.3 If $i_2 > N_{It}$, then go to step 3.7;

Set

$$\tilde{B}_0(t) = f(U^{(0)}(t)), a_i = a + \frac{b-a}{2}i, i = 0, 1, 2$$

step 3.4 Set $j = 1$;

step 3.5 If $j > n$, then go to step 3.6;

Set

$$D_j = \left\{ \left(a_i, I_G(\tilde{B}_{j-1}(a_i - t)F_j(t)P_j'(t) \exp(-\mu t), 0, a_i) + r_{j+1}(a_i) \right) : i = 0, 1, 2 \right\},$$

$$\tilde{B}_j(t) = \begin{cases} \tilde{B}_j(t) & t \leq a, \\ P(D_j) & a < t \leq b, \\ 0 & \text{Otherwise,} \end{cases}$$

where $P(D_j)$ is the interpolation polynomial that passes through the points D_j .

Set $j = j + 1$ and return to step 3.5;

step 3.6 Set

$$D_{n+1} = \left\{ \left(a_i, I_G(-\mu \left(U^{(0)}(t) \right)_1 - \tilde{B}_0(t) + \tilde{B}_n(t), 0, a_i) + r_{n+2}(a_i) \right) : i = 0, 1, 2 \right\},$$

$$D_{j+n+1} = \left\{ \left(a_i, I_G(\tilde{B}_{j-1}(a_i - t)F_j(t)P_j(t) \exp(-\mu t), 0, a_i) + r_{j+n+2}(a_i) \right) : i = 0, 1, 2 \right\} \quad j = 1, \dots, n,$$

$$U^{(0)}(t) = \begin{bmatrix} \begin{cases} (U^{(0)}(t))_1 & t \leq a, \\ P(D_{n+1}) & a < t \leq b, \\ 0 & \text{Otherwise.} \end{cases} \\ \begin{cases} (U^{(0)}(t))_2 & t \leq a, \\ P(D_{n+2}) & a < t \leq b, \\ 0 & \text{Otherwise.} \end{cases} \\ \vdots \\ \begin{cases} (U^{(0)}(t))_{n+1} & t \leq a, \\ P(D_{2n+1}) & a < t \leq b, \\ 0 & \text{Otherwise.} \end{cases} \end{bmatrix},$$

where $(U^{(0)}(t))_i$ means the i th component of $U^{(0)}(t)$.

Set $i_2 = i_2 + 1$ and return to step 3.3;

step 3.7 Set $a = b$,

$b = b + l$,

Set $i_1 = i_1 + 1$ and return to step 3.1;

step4. Print the column vector $U^{(0)}(t)$ as the approximation of $U(t) = (S(t), I_1(t), \dots, I_n(t))^T$, and end the algorithm.

2.2. Convergence Analysis. In accordance with the hypothesis H ,

$D_f = \{(S, I_1, \dots, I_n) \mid S \geq 0, (I_1, \dots, I_n) \in [0, \infty)^n\}$ is the domain of f . We suppose that f has a Lipschitz condition with Lipschitz constant L on D_f , i.e., there exists $L > 0$ such that

$$\|f(U) - f(V)\| \leq L\|U - V\|, \quad U, V \in D_f.$$

Suppose $i \in \{1, \dots, N_I\}$ and arbitrary and let $t \in J_i$ and arbitrary. By defining

$$U^{(m)}(t) := \begin{bmatrix} S^{(m)}(t) \\ I_1^{(m)}(t) \\ \vdots \\ I_n^{(m)}(t) \end{bmatrix}, \quad m = 0, 1, 2, \dots,$$

$$E^{(m)}(t) := U^{(m)}(t) - U^{(m-1)}(t), \quad m \in \mathbb{N}$$

$$e_m = \|E^{(m)}\|_{J_i} := \sup\{E^{(m)}(t) : t \in J_i\},$$

the incidence function B_0 has the following property

$$\begin{aligned} |B_0^{(m+1)}(t) - B_0^{(m)}(t)| &= \left| f\left(U^{(m)}(t)\right) - f\left(U^{(m-1)}(t)\right) \right| \\ &\leq L \|U^{(m)} - U^{(m-1)}\|_{J_i} = L \|E^{(m)}\|_{J_i} = Le_m. \end{aligned}$$

Since $t \in J_i$ and arbitrary, supremum property implies that

$$\|B_0^{(m+1)} - B_0^{(m)}\|_{J_i} \leq Le_m.$$

For the function B_1 we obtain

$$\begin{aligned} |B_1^{(m+1)}(t) - B_1^{(m)}(t)| &= \left| \int_{(i-1)l}^t \left(B_0^{(m+1)}(a) - B_0^{(m)}(a) \right) F_1(t-a) P_1'(t-a) e^{-\mu(t-a)} da \right| \\ &\leq Le_m M_1(l), \end{aligned}$$

where

$$M_1(l) := \sup_{t \in J_i} \left| \int_{(i-1)l}^t F_1(t-a) P_1'(t-a) e^{-\mu(t-a)} da \right| \rightarrow 0 \text{ as } l \rightarrow 0.$$

Since $t \in J_i$ and arbitrary, supremum property yields

$$\|B_1^{(m+1)} - B_1^{(m)}\|_{J_i} \leq Le_m M_1(l).$$

By the bounded induction, [20] (Chapter 5, Section 10, Problem2) we see that

$$\|B_j^{(m+1)} - B_j^{(m)}\|_{J_i} \leq Le_m M_1(l) \dots M_j(l), \quad j = 1, \dots, n,$$

where

$$M_j(l) := \sup_{t \in J_i} \left| \int_{(i-1)l}^t F_j(t-a) P_j'(t-a) e^{-\mu(t-a)} da \right| \rightarrow 0 \text{ as } l \rightarrow 0.$$

And hence by defining

$$(2.3) \quad \alpha_j(l) = L \begin{cases} 1 & j = 0, \\ M_1(l) \dots M_j(l) & j = 1, \dots, n, \end{cases}$$

we obtain

$$(2.4) \quad \left\| B_j^{(m+1)} - B_j^{(m)} \right\|_{J_i} \leq \alpha_j(l) e_m, \quad j = 0, 1, \dots, n,$$

Equations (2.1) and (2.4) implies that

$$\begin{aligned} & \left| \left(E^{(m+1)}(t) \right)_1 \right| = \left| S^{(m+1)}(t) - S^{(m)}(t) \right| \\ & \left| \mu \int_{(i-1)l}^t \left(S^{(m-1)}(a) - S^{(m)}(a) \right) da + \int_{(i-1)l}^t \left(B_0^{(m)}(a) - B_0^{(m+1)}(a) \right) da \right. \\ & \quad \left. + \int_{(i-1)l}^t \left(B_n^{(m+1)}(a) - B_n^{(m)}(a) \right) da \right| \\ & \leq \mu l \left\| S^{(m)} - S^{(m-1)} \right\|_{J_i} + l \left\| B_0^{(m+1)} - B_0^{(m)} \right\|_{J_i} + l \left\| B_n^{(m+1)} - B_n^{(m)} \right\|_{J_i} \\ (2.5) \quad & \leq (\mu l + Ll + LlM_1(l) \dots M_n(l)) \left\| E^{(m)} \right\|_{J_i} =: \beta_0(l) e_m \rightarrow 0 \text{ as } l \rightarrow 0. \end{aligned}$$

Now suppose $j \in \{1, \dots, n\}$ and arbitrary. Equations (2.2) and (2.4) forces to

$$\begin{aligned} & \left| \left(E^{(m+1)}(t) \right)_{j+1} \right| = \left| I_j^{(m+1)}(t) - I_j^{(m)}(t) \right| \\ & \left| \int_{(i-1)l}^t \left(B_{j-1}^{(m+1)}(a) - B_{j-1}^{(m)}(a) \right) F_j(t-a) P_j(t-a) e^{-\mu(t-a)} da \right| \\ (2.6) \quad & \leq \alpha_{j-1}(l) e_m \left| \int_{(i-1)l}^t F_j(t-a) P_j(t-a) e^{-\mu(t-a)} da \right| \leq \beta_j(l) e_m, \end{aligned}$$

where

$$\beta_j(l) := \alpha_{j-1}(l) \sup_{t \in J_i} \left| \int_{(i-1)l}^t F_j(t-a) P_j(t-a) e^{-\mu(t-a)} da \right| \rightarrow 0 \text{ as } l \rightarrow 0.$$

By defining $\beta(l) := \max \{ \beta_j(l) : j = 1, \dots, n \}$ and applying equations (2.5)-(2.6), we obtain

$$(2.7) \quad e_{m+1} = \left\| E^{(m+1)} \right\|_{J_i} \leq \beta(l) e_m \rightarrow 0 \text{ as } l \rightarrow 0.$$

Since $\lim_{l \rightarrow 0} \beta(l) = 0$, then there exists $l > 0$ such that $0 < \beta(l) < 1$. This value of l and using an induction on m , implies that

$$(2.8) \quad e_m \leq (\beta(l))^m e_0 \rightarrow 0 \text{ as } l \rightarrow 0.$$

The equation (2.8) not only shows the convergence of the method, but furthermore it shows the rate of convergence.

3. Conclusions

3.1. A sample problem with one infection period. For constitution of an example we suppose

$$(3.1) \quad f(S, I_1, \dots, I_n) = S \sum_{j=1}^n I_j(t).$$

For $n = 1$ and with the notations of the Sec. 2.1 we have

$$X(t) = \begin{bmatrix} B_0(t) \\ B_1(t) \\ S(t) \\ I_1(t) \end{bmatrix}, \quad r(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \\ r_4(t) \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-\mu t} \check{B}_1(t) \\ \check{S} + \Lambda t \\ e^{-\mu t} \check{I}_1(t) \end{bmatrix},$$

A simple choice for $S(t)$ can be written in the form $S(t) = \check{S}e^{-\gamma t}$, where $\gamma > 0$ is a constant. In this case the system (2.2) is as follow

$$(3.2) \quad \begin{cases} B_0(t) = \check{S}e^{-\gamma t} I_1(t), \\ B_1(t) = \int_0^t B_0(t-a) F_1(a) P_1'(a) e^{-\mu a} da + e^{-\mu t} \check{B}_1(t), \\ \check{S}e^{-\gamma t} = \check{S} + \Lambda t + \int_0^t \left(-\mu \check{S}e^{-\gamma a} - B_0(a) + B_1(a) \right) da, \\ I_1(t) = \int_0^t B_0(t-a) F_1(a) P_1(a) e^{-\mu a} da + e^{-\mu t} \check{I}_1(t), \end{cases}$$

By giving known values (usually exponential of negative variable) for functions P_1, F_1 and I_1 we obtain $B_0(t)$ from the first equation and then $\check{I}_1(t)$ obtains from the last equation. Now the third equation is an equation with unknown function B_1 . We take Laplace transform from this equation and obtain $\mathcal{L}\{B_1(t)\} = \hat{B}_1(\omega)$. Hence $B_1(t) = \mathcal{L}^{-1}\{\hat{B}_1(\omega)\}$ is obtainable. Finally the second equation gives a known value for $\check{B}_1(t)$.

Example 3.1. For the known values

$$\begin{aligned} \mu &= \frac{1}{2}, \check{S} = e^{\frac{1}{2}}, \Lambda = \frac{1}{2}, F_1(t) = e^{\frac{2(1-t)}{3}}, P_1(t) = e^{\frac{1-t}{4}} \\ \check{B}_1(t) &= e^{\frac{t}{2}} \left(-\frac{1}{2} + e^{-\frac{5}{6}(-1+t)} + \frac{3}{7} e^{\frac{7}{4} - \frac{17t}{12}} \left(-1 + e^{\frac{17t}{12}} \right) \right), \\ \check{I}_1(t) &= \frac{12}{7} e^{\frac{7}{4} - \frac{11t}{12}} - \frac{12}{7} e^{\frac{7}{4} - \frac{t}{3}} + e^{-\frac{2+t}{6}}, \end{aligned}$$

the system

$$(3.3) \quad \begin{cases} B_0(t) = S(t) I_1(t), \\ B_1(t) = \int_0^t B_0(t-a) F_1(a) P_1'(a) e^{-\mu a} da + e^{-\mu t} \check{B}_1(t), \\ S(t) = \check{S} + \Lambda t + \int_0^t \left(-\mu S(a) - B_0(a) + B_1(a) \right) da, \\ I_1(t) = \int_0^t B_0(t-a) F_1(a) P_1(a) e^{-\mu a} da + e^{-\mu t} \check{I}_1(t), \end{cases}$$

has the exact solution

$$X(t) = \begin{bmatrix} B_0(t) \\ B_1(t) \\ S(t) \\ I_1(t) \end{bmatrix} = \begin{bmatrix} e^{\frac{5(1-t)}{6}} \\ e^{\frac{5(1-t)}{6}} - \frac{1}{2} \\ e^{\frac{1-t}{2}} \\ e^{\frac{1-t}{3}} \end{bmatrix}.$$

We solve this system on $t \in [0, 10]$, by the described algorithm in Sec. 2.1 with $l = 0.1$, $N_I = 100$, $N_{It} = 7$, $N_G = 16$. Figure 1 shows a comparison between $S(t)$ and $\tilde{S}(t)$ as functions of t for Example 3.1. Similarly Figure 2 shows a comparison between $I_1(t)$ and $\tilde{I}_1(t)$ as functions of t for Example 3.1. $S(t), I_1(t)$ are the exact solutions and $\tilde{S}(t), \tilde{I}_1(t)$ are the approximated solutions by the proposed method. Table 1 shows absolute errors of $\tilde{S}$ and $\tilde{I}_1$ in the nodal points $t_i = 0.5i$, $i = 0, 1, \dots, 20$.

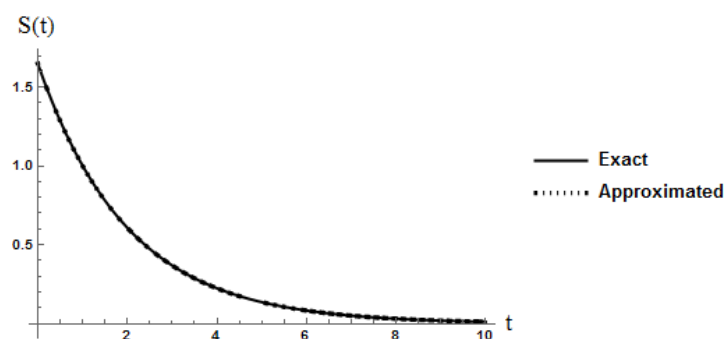


FIGURE 1. Variations of $S(t)$ and $\tilde{S}(t)$ as functions of t for Example 3.1

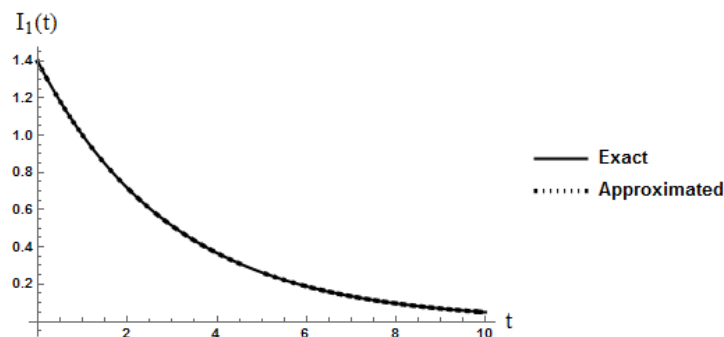


FIGURE 2. Variations of $I_1(t)$ and $\tilde{I}_1(t)$ as functions of t for Example 3.1

TABLE 1. Absolute errors of $\tilde{I}_1$ and $\tilde{S}$ at $t_i = 0.5i$, $i = 0, 1, \dots, 20$ for Example 3.1

i	Errors of $\tilde{I}_1$	Errors of $\tilde{S}$
0	0.0	0.0
1	4.29×10^{-7}	3.01×10^{-7}
2	7.03×10^{-7}	3.01×10^{-7}
3	2.87×10^{-7}	7.53×10^{-7}
4	3.41×10^{-7}	3.34×10^{-7}
5	2.43×10^{-7}	1.26×10^{-7}
6	6.04×10^{-7}	2.11×10^{-7}
7	6.07×10^{-7}	2.85×10^{-7}
8	9.06×10^{-7}	5.49×10^{-7}
9	1.45×10^{-7}	1.19×10^{-6}
10	1.60×10^{-7}	2.79×10^{-6}
11	7.89×10^{-7}	1.84×10^{-6}
12	2.12×10^{-7}	2.06×10^{-6}
13	5.82×10^{-7}	1.56×10^{-6}
14	3.57×10^{-7}	2.55×10^{-6}
15	5.50×10^{-7}	8.36×10^{-6}
16	4.77×10^{-7}	7.29×10^{-6}
17	5.27×10^{-7}	4.37×10^{-6}
18	2.00×10^{-7}	3.78×10^{-6}
19	2.83×10^{-7}	4.18×10^{-6}
20	8.90×10^{-7}	6.71×10^{-6}

3.2. A sample problem with two infection period. In this section we sketch a sample problem with $n = 2$. Suppose f is given by (3.1). In accordance with the notations of the Sec. 2.1 we have

$$X(t) = \begin{bmatrix} B_0(t) \\ B_1(t) \\ B_2(t) \\ S(t) \\ I_1(t) \\ I_2(t) \end{bmatrix}, \quad r(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \\ r_4(t) \\ r_5(t) \\ r_6(t) \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-\mu t} \check{B}_1(t) \\ e^{-\mu t} \check{B}_2(t) \\ \check{S} + \Lambda t \\ e^{-\mu t} \check{I}_1(t) \\ e^{-\mu t} \check{I}_2(t) \end{bmatrix},$$

In accordance with the Sec. 3.1 we set $S(t) = \check{S}e^{-\gamma t}$, for some $\gamma > 0$. Hence in this case the system (2.2) is as follow

$$(3.4) \quad \begin{cases} B_0(t) = \check{S}e^{-\gamma t} (I_1(t) + I_2(t)), \\ B_1(t) = \int_0^t B_0(t-a)F_1(a)P_1'(a)e^{-\mu a}da + e^{-\mu t}\check{B}_1(t), \\ B_2(t) = \int_0^t B_1(t-a)F_2(a)P_2'(a)e^{-\mu a}da + e^{-\mu t}\check{B}_1(t), \\ \check{S}e^{-\gamma t} = \check{S} + \Lambda t + \int_0^t \left(-\mu\check{S}e^{-\gamma a} - B_0(a) + B_2(a) \right) da, \\ I_1(t) = \int_0^t B_0(t-a)F_1(a)P_1(a)e^{-\mu a}da + e^{-\mu t}\check{I}_1(t) \\ I_2(t) = \int_0^t B_1(t-a)F_2(a)P_2(a)e^{-\mu a}da + e^{-\mu t}\check{I}_2(t), \end{cases}$$

By giving the known values (usually exponential functions of negative variable) for functions $P_i, F_i, I_i, i = 1, 2$, we obtain $B_0(t)$ from the first equation and then $\check{I}_1(t)$ obtains from the

fifth equation. Now the fourth equation is an equation with unknown function B_2 . We take Laplace transform from this equation and obtain $\mathcal{L}\{B_2(t)\} = \widehat{B}_2(\omega)$. Then $B_2(t)$ is obtained from $B_2(t) = \mathcal{L}^{-1}\widehat{B}_2(\omega)$. By introducing the known value (usually exponential function of negative variable) for function $B_1(t)$, unknown function $\check{B}_1(t)$ is obtained from the second equation of (3.4). By substitution of all known functions in the third and sixth equation of (3.4), we obtain the rest unknown functions $\check{B}_2(t)$ and $\check{I}_2(t)$ respectively.

Example 3.2. For the known data

$$\begin{aligned} \mu &= \frac{1}{2}, \check{S} = e^{\frac{1}{2}}, \Lambda = \frac{1}{2}, F_1(t) = e^{-\frac{t}{2}}, P_1(t) = e^{-t}, F_2(t) = e^{-\frac{t}{2}}, P_2(t) = e^{-t}, \\ \check{B}_1(t) &= e^{\frac{t}{2}} \left(e^{\frac{1}{2}-\frac{t}{2}} - \frac{2}{15}e^{\frac{7}{6}-2t} \left(9 - 9e^{\frac{5t}{6}} - 10e^{\frac{1}{12}} \left(-1 + e^{\frac{3t}{4}} \right) \right) \right), \\ \check{B}_2(t) &= e^{\frac{t}{2}} \left(-\frac{1}{2} + e^{-\frac{5}{4}(-1+t)} + e^{-\frac{7}{6}(-1+t)} + \frac{2}{3}e^{\frac{1}{2}-2t} \left(-1 + e^{\frac{3t}{2}} \right) \right), \\ \check{I}_1(t) &= e^{\frac{t}{2}} \left(e^{-\frac{2}{3}(-1+t)} - \frac{2}{15}e^{\frac{7}{6}-2t} \left(10e^{\frac{1}{12}} \left(-1 + e^{\frac{3t}{4}} \right) + 9 \left(-1 + e^{\frac{5t}{6}} \right) \right) \right), \\ \check{I}_2(t) &= e^{\frac{t}{2}} \left(e^{\frac{3}{4}-\frac{3t}{4}} - \frac{2}{3}e^{\frac{1}{2}-2t} \left(-1 + e^{\frac{3t}{2}} \right) \right), \end{aligned}$$

the system

$$(3.5) \quad \begin{cases} B_0(t) = S(t) (I_1(t) + I_2(t)), \\ B_1(t) = \int_0^t B_0(t-a)F_1(a)P_1'(a)e^{-\mu a}da + e^{-\mu t}\check{B}_1(t), \\ B_2(t) = \int_0^t B_1(t-a)F_2(a)P_2'(a)e^{-\mu a}da + e^{-\mu t}\check{B}_2(t), \\ S(t) = \check{S} + \Lambda t + \int_0^t \left(-\mu S(a) - B_0(a) + B_2(a) \right) da, \\ I_1(t) = \int_0^t B_0(t-a)F_1(a)P_1(a)e^{-\mu a}da + e^{-\mu t}\check{I}_1(t), \\ I_2(t) = \int_0^t B_1(t-a)F_2(a)P_2(a)e^{-\mu a}da + e^{-\mu t}\check{I}_2(t), \end{cases}$$

has the exact solution

$$X(t) = \begin{bmatrix} B_0(t) \\ B_1(t) \\ B_2(t) \\ S(t) \\ I_1(t) \\ I_2(t) \end{bmatrix} = \begin{bmatrix} e^{-\frac{5}{4}(-1+t)} + e^{-\frac{7}{6}(-1+t)} \\ e^{\frac{1}{2}-\frac{t}{2}} \\ e^{-\frac{5}{4}(-1+t)} + e^{-\frac{7}{6}(-1+t)} - \frac{1}{2} \\ e^{\frac{1}{2}-\frac{t}{2}} \\ e^{\frac{2(1-t)}{3}} \\ e^{\frac{3(1-t)}{4}}, \end{bmatrix},$$

We solve this system on $t \in [0, 10]$, by the described algorithm in Sec. 2.1 with $l = 0.1$, $N_I = 100$, $N_{It} = 7$, $N_G = 16$. Figure 1 shows a comparison between $S(t)$ and $\tilde{S}(t)$ as functions of t for Example 3.2. Similarly Figures 2 and 3 show a comparison between $I_i(t)$ and $\tilde{I}_i(t)$, $i = 1, 2$ as functions of t for Example 3.2. $\tilde{S}(t)$, $\tilde{I}_i(t)$, $i = 1, 2$ are the exact solutions and $\tilde{S}(t)$, $\tilde{I}_i(t)$, $i = 1, 2$ are the approximated solutions by the proposed method. Table 2 shows absolute errors of $\tilde{S}$ and $\tilde{I}_i$, $i = 1, 2$ in the nodal points $t_i = 0.5i$, $i = 0, 1, \dots, 20$.

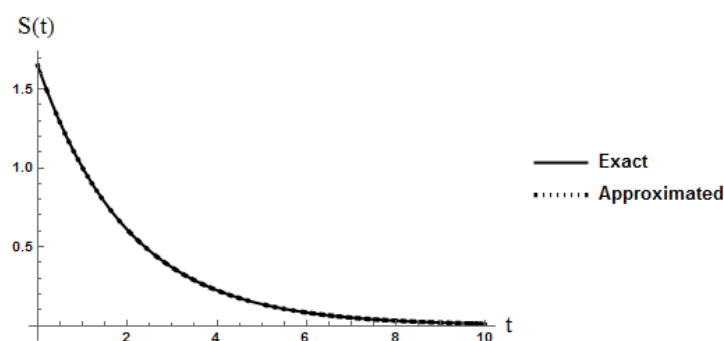


FIGURE 3. Variations of $S(t)$ and $\tilde{S}(t)$ as functions of t for Example 3.2

TABLE 2. Absolute errors of $\tilde{I}_1$ and $\tilde{S}$ at $t_i = 0.5i$, $i = 0, 1, \dots, 20$ for Example 3.2

i	Errors of $\tilde{I}_1$	Errors of $\tilde{I}_2$	Errors of $\tilde{S}$
0	0.0	0.0	0.0
1	3.96×10^{-7}	2.92×10^{-7}	1.15×10^{-6}
2	9.04×10^{-7}	2.85×10^{-7}	4.09×10^{-6}
3	2.33×10^{-6}	1.75×10^{-7}	3.82×10^{-7}
4	2.06×10^{-6}	2.49×10^{-7}	8.99×10^{-7}
5	9.28×10^{-7}	1.37×10^{-7}	2.10×10^{-5}
6	1.40×10^{-7}	9.09×10^{-7}	5.74×10^{-6}
7	8.03×10^{-7}	1.41×10^{-7}	4.28×10^{-6}
8	1.26×10^{-6}	1.46×10^{-7}	4.66×10^{-6}
9	1.23×10^{-7}	7.56×10^{-8}	1.87×10^{-5}
10	4.49×10^{-7}	1.37×10^{-7}	1.98×10^{-5}
11	6.39×10^{-7}	2.54×10^{-7}	1.28×10^{-6}
12	1.84×10^{-7}	1.36×10^{-7}	3.31×10^{-5}
13	4.94×10^{-7}	4.73×10^{-8}	1.53×10^{-5}
14	1.95×10^{-7}	9.44×10^{-8}	4.38×10^{-6}
15	1.37×10^{-7}	3.70×10^{-8}	4.83×10^{-5}
16	2.43×10^{-7}	9.05×10^{-9}	3.82×10^{-5}
17	1.49×10^{-7}	9.77×10^{-9}	1.37×10^{-5}
18	1.27×10^{-9}	1.26×10^{-8}	1.33×10^{-5}
19	3.47×10^{-8}	2.38×10^{-8}	2.84×10^{-5}
20	1.24×10^{-7}	1.50×10^{-8}	4.10×10^{-5}

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