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GEODESIC VECTORS OF SQUARE METRICS ON 5- DIMENSIONAL GENERALIZED SYMMETRIC SPACES

DARIUSH LATIFI* AND MILAD L. ZEINALI

Abstract. In this paper, we consider the (α, β)-metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ along with the function ϕ with the definition of $\phi(s) = 1 + 2s + s^2$, which is known as a square metric, on 5-dimensional generalized symmetric spaces. Then we investigate and study the homogeneous geodesics of 5-dimensional generalized symmetric spaces equipped with a left invariant square metric. We also obtain and categorize the homogeneous geodesics of these spaces in some special cases, which are of type (2), (3) and (7). Also we show that for a 5-dimensional generalized symmetric space of type (2) and (7) equipped with a left invariant square metric, the geodesic vectors of (M, F) are the same as the geodesic vectors of $(M, \tilde{a})$ and vice versa.

1. Introduction

Let M be a smooth n-dimensional C^∞ manifold and TM be its tangent bundle. A Finsler metric on M is a non-negative function $F : TM \to \mathbb{R}$ with the following properties [2]:

(1) F is smooth on the slit tangent bundle $TM^0 := TM \backslash \{0\}$.
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$ and $\lambda > 0$.

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(3) The following bilinear symmetric form $g_y : T_xM \times T_xM \longrightarrow \mathbb{R}$ is positive definite

$$g_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(x, y + su + tv)|_{s=t=0}.$$

The (α,β)-metrics form an important class of Finsler metrics appearing iteratively in formulating physics, mechanics, seismology, biology, control theory, etc (see [1]). For the first time, in 1972, Matsumoto had introduced the concept of (α,β)-metric in Finsler geometry [14]. (α,β)-metrics are the generalizations of the Randers metric, introduced by G. Randers [16]. An (α,β)-metric is a Finsler metric of the form $F = \alpha\varphi(s)$, $s = \frac{\beta}{\alpha}$ where $\alpha = \sqrt{\tilde{a}_{ij}(x)y^iy^j}$ is induced by a Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ on a connected smooth n-dimensional manifold M and $\beta = b_i(x)y^i$ is a 1-form on M. The important types of (α,β)-metrics happen when we have

$$\varphi(s) = 1 + 2s + s^2.$$

In this case we have

$$F = \frac{(\alpha+\beta)^2}{\alpha},$$

which called a square space.

The Riemannian metric $\widetilde{a}$ induces an inner product on any cotangent space T_x^*M such that

$$\langle dx^i(x), dx^j(x)\rangle = \widetilde{a}^{ij}(x).$$

The induced inner product on T_x^*M induced a linear isomorphism between T_x^*M and T_xM. Then the 1-form β corresponds to a vector field $\widetilde{X}$ on M such that

$$\widetilde{a}\left(y, \widetilde{X}\left(x\right)\right) = \beta\left(x,y\right).$$

Also, we have

$$\|\beta(x)\|_\alpha = \|\widetilde{X}(x)\|_\alpha.$$

Therefore we can write (α,β)-metrics as follows:

$$F\left(x,y\right) = \alpha\left(x,y\right)\varphi\left(\frac{\widetilde{a}(\widetilde{X}\left(x\right), y)}{\alpha(x,y)}\right), \tag{1.1}$$

where for any $x \in M$,

$$\sqrt{\widetilde{a}\left(\widetilde{X}\left(x\right), \widetilde{X}\left(x\right)\right)} = \|\widetilde{X}(x)\|_\alpha < b_0.$$

So any square metrics can be written as

$$F(x,y) = \frac{\left(\alpha(x,y) + \tilde{a}(\tilde{X}(x), y)\right)^2}{\alpha(x,y)}.$$

Let π^*TM be the pull-back of the tangent bundle TM by $\pi : TM^0 \to M$. Unlike the Levi-Civita connection in Riemannian geometry, there is no unique natural connection in the Finsler case. Among

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these connections on π^*TM, we choose the Chern connection whose coefficients are denoted by Γ^i_{jk} [2]. This connection is almost g-compatible and has no torsion. Since, in general, the Chern connection coefficients Γ^i_{jk} in natural coordinates have a directional dependence, we must define a fixed reference vector.

Let $\sigma(t)$ be a smooth regular curve in M, with velocity field T. Let $W(t) := W^i(t)\frac{\partial}{\partial x^i}$ be a vector field along σ. The expression

$$\Big[\frac{dW^i}{dt} + W^jT^k(\Gamma^i_{jk})_{(\sigma,T)}\Big]\frac{\partial}{\partial x^i}|_{\sigma(t)},$$

would have defined the covariant derivative D_TW with reference vector T. A curve $\sigma(t)$, with velocity $T = \dot{\sigma}(t)$ is a Finslerian geodesic if

$$D_T\Big[\frac{T}{F(T)}\Big] = 0, \quad \text{with reference vector } T,$$

that the constant speed geodesics are precisely the solution of

$$D_TT = 0, \quad \text{with reference vector } T.$$

Definition 1.1. *A Finsler space (M, F) is called a homogeneous Finsler space if the group of isometries of (M, F), $I(M, F)$ acts transitively on M.*

Definition 1.2. *Let $(G/H, F)$ be a homogeneous Finsler space and e be the identity of G. A non-zero vector $X \in \mathfrak{g}$ is called a geodesic vector if the curve $exp(tX).eH$ is a geodesic of $(G/H, F)$.*

Definition 1.3. *A geodesic $\sigma(t)$ through the $o \in M$ is called a homogeneous geodesic if it is an orbit of a one-parameter subgroup of G, that is*

$$\sigma(t) = exp(tZ)(o), \quad t \in \mathbb{R},$$

where Z is a nonzero vector of $\mathfrak{g}$.

In 1965, Hermann showed that homogeneous geodesics which are orbits of a given one-parameter group of isometries $A(t)$ correspond to the critical points of the norm of Killing vector field X which generates $A(t)$. Kostant, Vinberg, Kowalski and Vanhecke found a simple condition that the orbit

$$\sigma(t) = A(t)(eH),$$

of an one-parameter subgroup $A(t) = exp(tX) \subset G$ of the isometry group G of a homogeneous Riemannian manifold $M = G/H$, is a geodesic [6, 8, 18].

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $\tilde{a}$ be a left invariant Riemannian metric on G. In [8], the author proved that the following result for Riemannian metrics.

Lemma 1.4. *a vector $Y \in \mathfrak{g}$ is a geodesic vector if and only if*

$$\tilde{a}(Y, [Y, Z]) = 0, \quad \forall Z \in \mathfrak{g}. \tag{1.2}$$

In [10], Latifi proved the following result that gives a criterion for a non-zero vector to be a geodesic vector in a homogeneous Finsler space.

Lemma 1.5. *A non-zero vector $Y \in \mathfrak{g}$ is a geodesic vector if and only if*

$$g_{Y_{\mathfrak{m}}}(Y_{\mathfrak{m}}, [Y, Z]_{\mathfrak{m}}) = 0, \quad \forall Z \in \mathfrak{g}. \tag{1.3}$$

We now recall some basic facts about generalized symmetric spaces. A generalized symmetric space is a connected Riemannian manifold $(M, \tilde{a})$ admitting a regular s-structure, that is, a family

$$\{s_x : x \in M\},$$

of symmetries on M, such that

$$s_x \circ s_y = s_z \circ s_x, \quad z = s_x(y),$$

for every points $x, y, \in M$. As it is well-known, every generalized symmetric space is a homogeneous Riemannian space G/H. An s-structure $\{s_x : x \in M\}$ is said to be of order $k \geq 2$ if $(s_x)^k = id$ for all $x \in M$ and $(s_x)^i \neq id$ for $i < k$. A Riemannian manifold $(M, \tilde{a})$ is said to be k-symmetric if it admits a regular s-structure of order k. Each generalized symmetric space is k-symmetric for some k. The order of a generalized symmetric space is the least integer k such that $(M, \tilde{a})$ is k-symmetric. Low dimensional generalized symmetric spaces have been classified in [7].

2. Main Results

A five-dimensional generalized symmetric space M of type **(2)** is $\mathbb{R}^5(x, y, z, w, t)$, equipped with the Riemannian metric

$$\begin{aligned} \widetilde{a} = {} & e^{-2\lambda_1 q}dx^2 + e^{2\lambda_1 q}dy^2 + e^{-2\lambda_2 q}dz^2 + e^{2\lambda_2 q}dw^2 + dq^2 \\ & + 2\alpha[e^{-(\lambda_1+\lambda_2)q}dxdz + e^{(\lambda_1+\lambda_2)q}dydw] \\ & + 2\beta[e^{(\lambda_1-\lambda_2)q}dydz - e^{(\lambda_2-\lambda_1)q}dxdw], \end{aligned}$$

where either $(2a)\lambda_1 > \lambda_2 > 0, \alpha^2 + \beta^2 < 1$, or $(2b)\lambda_1 = \lambda_2 > 0$, $\alpha = 0$ and $0 \leq \beta < 1$, or $(2c)\lambda_1 < 0, \lambda_2 = 0, \alpha = 0$ and $0 < \beta < 1$. As homogeneous space, $M = G/H$, where G is the group of all matrices of the form

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$$\begin{bmatrix} e^{\lambda_1 q} & 0 & 0 & 0 & x \\ 0 & e^{-\lambda_1 q} & 0 & 0 & y \\ 0 & 0 & e^{\lambda_2 q} & 0 & z \\ 0 & 0 & 0 & e^{-\lambda_2 q} & w \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

For type (2) of these spaces we prove that the following Theorem for types (2a), (2b) and (2c):

Theorem 2.1. *Let (M, F) be a 5-dimensional generalized symmetric space of type **(2a)** equipped with a left invariant square metric defined by the Riemannian metric $\tilde{a}$ and vector field $X = x_5 e_5$. Then $Y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if Y is a geodesic vector of $(M, \tilde{a})$.*

Theorem 2.2. *Let (M, F) be a 5-dimensional generalized symmetric space of type **(2b)** equipped with a left invariant square metric defined by the Riemannian metric $\tilde{a}$ and vector field $X = x_5 e_5$. Then $Y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if Y is a geodesic vector of $(M, \tilde{a})$.*

Theorem 2.3. *Let (M, F) be a 5-dimensional generalized symmetric space of type **(2c)** equipped with a left invariant square metric defined by the Riemannian metric $\tilde{a}$ and vector field $X = x_5 e_5$. Then $Y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if Y is a geodesic vector of $(M, \tilde{a})$.*

As homogeneous spaces, five-dimensional generalized symmetric spaces M of type **(7)** are real matrix groups

$$\begin{bmatrix} e^{\lambda t} & 0 & 0 & 0 & x \\ 0 & e^{-\lambda t} & 0 & 0 & y \\ te^{\lambda t} & 0 & e^{\lambda t} & 0 & u \\ 0 & -te^{-\lambda t} & 0 & e^{-\lambda t} & v \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

M is also $\mathbb{R}^5(x, y, u, v, t)$, equipped with a Riemannian metric

$$\widetilde{a} = dt^2 + e^{-2\lambda t}(tdx - du)^2 + e^{2\lambda t}(tdy + dv)^2 + a^2(e^{-2\lambda t}dx^2 + e^{2\lambda t}dy^2) + 2\gamma(dydu - dxdv),$$

where $a > 0, \lambda \geq 0, \lambda, a, \gamma \in \mathbb{R}$ and $\gamma^2 < a^2$. We have two case for type (7) as follows:

- (7a) $\lambda \neq 0$,
- (7b) $\lambda = 0$.

For type (7) of these spaces we prove that the following Theorem for types (7a) and (7b):

Theorem 2.4. *Let (M, F) be a 5-dimensional generalized symmetric space of type **(7a)** equipped with a left invariant square metric defined by the Riemannian metric $\tilde{a}$ and vector field $X = x_5 e_5$. Then $Y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if Y is a geodesic vector of $(M, \tilde{a})$.*

Theorem 2.5. *Let (M, F) be a 5-dimensional generalized symmetric space of type* **(7b)** *equipped with a left invariant square metric defined by the Riemannian metric $\tilde{a}$ and vector field $X = x_5 e_5$. Then $Y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if Y is a geodesic vector of $(M, \tilde{a})$.*

Also we have the following corollary for type **(3)**:

Corollary 2.6. *Let (M, F) be a five-dimensional generalized symmetric spaces of type 3 equipped with a left invariant square metric defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then geodesic vectors depending only on $x_1, x_2, x_3, x_4, x_5, t$ and y_5.*

We also categorize the geodesic vectors of these spaces. For example, we show that Y is a geodesic vector of a generalized symmetric space of type **(2a)** equipped with a left invariant square metric defined by the Riemannian metric $\tilde{a}$ and vector field X if and only if

$$Y = -\frac{1}{A}\sum_{i=1}^{4} x_i e_i + y_5 e_5, \quad A \neq 0,$$

or $Y = \sum_{i=1}^{4} y_i e_i, \quad A \neq 0$, and the following equation is hold:

$$\begin{aligned} & y_1\big(-\lambda_1 y_1\big) + y_2\Big(-\lambda_2 y_2 + \frac{\alpha(\lambda_1-\lambda_2)}{\sqrt{1-\alpha^2}} y_1\Big) \\ & + y_3\left(\lambda_1 y_3 + \frac{\beta}{\sqrt{1-\alpha^2-\beta^2}}\Big((\lambda_1+\lambda_2)y_2 + \frac{\alpha(\lambda_2-\lambda_1)}{\sqrt{1-\alpha^2}} y_1\Big)\right) \\ & + y_4\left(\lambda_2 y_4 - \frac{\beta(\lambda_1+\lambda_2)}{\sqrt{1-\alpha^2-\beta^2}} y_1 - \frac{\alpha(\lambda_1-\lambda_2)}{\sqrt{1-\alpha^2}} y_3 - \frac{\alpha\beta(\lambda_1-\lambda_2)}{\sqrt{1-\alpha^2}\sqrt{1-\alpha^2-\beta^2}} y_2\right) = 0. \end{aligned}$$

Also we show that Y is a geodesic vector of a generalized symmetric space of type **(2b)** equipped with a left invariant square metric defined by the Riemannian metric $\tilde{a}$ and vector field X if and only if

$$Y = -\frac{1}{A}\sum_{i=1}^{4} x_i E_i + y_5 E_5 \;\text{ or }\; Y = \sum_{i=1}^{4} y_i E_i + tT, \quad A \neq 0,$$

or $Y = \sum_{i=1}^{4} y_i e_i, \quad A \neq 0$, and the following equation is hold:

$$y_1^2 + y_2^2 = y_3\Big(y_3 + \frac{2\beta}{\sqrt{1-\beta^2}} y_2\Big) + y_4\Big(y_4 - \frac{2\beta}{\sqrt{1-\beta^2}} y_1\Big).$$

3. Summary of Proofs/Conclusions

Conclusion of this article: In this paper, we consider five dimensional generalized symmetric spaces equipped with an invariant square metric. Square metrics are the important types of (α, β)-metrics. For a square metric we have $\varphi(s) = 1 + 2s + s^2$. In this case we have $F = \frac{(\alpha+\beta)^2}{\alpha}$. The study

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of homogeneous geodesics in these homogeneous spaces is of particular interest. For 5-dimensional generalized symmetric space of type (2) and (7) equipped with an invariant square metric, the geodesic vectors of (M, F) are the same as the geodesic vectors of $(M, \tilde{a})$ and vice versa and the geodesic vectors of type (3) of these spaces depending only on $x_1, x_2, x_3, x_4, x_5, t$ and y_5.

References

[1] P. L. Antonelli, A. Bona and M. A. Slawinski, Seismic rays as Finsler geodesics, *Nonlinear Anal. Real World Appl.*, **4** no. 5 (2003) 711–722.

[2] D. Bao, S. S. Chern and Z. Shen, *An introduction to Riemann-Finsler geometry*, Springer-Verlag, New York, 2000.

[3] G. Calvaruso and R. A. Marinosci, Homogeneous geodesics in five-dimensional generalized symmetric spaces, *Balkan J. Geom. Appl.*, **8** no. 1 (2003) 1–19.

[4] M. Ebrahimi and D. Latifi, Geodesic vectors of Randers metric on generalized symmetric spaces, *Glob. J. Adv. Res. Class. Mod. Geom.*, **2** no. 2 (2021) 153–165.

[5] P. Habibi, Geodesic vectors of invariant square metrics on nilpotent Lie groups of dimension five, *J. Finsler Geom. Appl.*, **2** no.1 (2021) 131–140.

[6] B. Kostant, *Holonomy and Lie algebra of motions in Riemannian manifolds*, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[7] O. Kowalski, *Generalized symmetric spaces Lecture Notes in Mathematics*, Springer-Verlag, Berlin, New York, 1980.

[8] O. Kowalski and L. Vanhecke, Riemannian manifolds with homogeneous geodesics, *Boll. Un. Mat. Ital. B (7)*, **5** no. 1 (1991) 189–246.

[9] O. Kowalski and J. Szenthe, On the existence of homogeneous geodesics in homogeneous Riemannian manifolds, *Geom. Dedicata*, **81** no. 1-3 (2000) 209–214.

[10] D. Latifi, Homogeneous geodesics in homogeneous Finsler spaces, *J. Geom. Phys.*, **57** no. 5 (2007) 1421–1433.

[11] D. Latifi and M. Toomanian, Invariant naturally reductive Randers metrics on homogeneous spaces, *Math. Sci. (Springer)*, **6** (2012) 5 pp.

[12] D. Latifi, Bi-invariant (α, β)-metrics on Lie groups, *Acta Univ. Apulensis Math. Inform.*, **65** (2021) 121–131.

[13] D. Latifi, On generalized symmetric square metrics, *Acta Univ. Apulensis Math. Inform.*, **68** (2021) 63–70.

[14] M. Matsumoto, On C-reducible Finsler spaces, *Tensor (N.S.)*, **24** (1972) 29–37.

[15] M. Parhizkar and H. R. Salimi Moghaddam, Naturally reductive homogeneous (α, β)-metric spaces, *Arch. Math. (Brno)*, **57** no. 1 (2021) 1–11.

[16] G. Randers, On an asymmetric metric in the four-space of general relativity, *Phys. Rev. (2)*, **59** (1941) 195–199.

[17] H. R. Salimi Moghadam, A class of generalized symmetric Finsler spaces, *Journal of Mathematics and Society*, **5** no. 3 (2020) 15–22. [In Persian]

[18] E. B. Vinberg, Invariant linear connection in a homogeneous manifold, *Trudy Moskov. Mat. Obšč.*, **9** (1960) 191–210.

[19] Z. Yan and L. Huang, On the existence of homogeneous geodesic in homogeneous Finsler spaces, *J. Geom. Phys.*, **124** (2018) 264–267.

[20] M. L. Zeinali, On generalized symmetric Finsler spaces with some special (α, β)-metrics, *J. Finsler Geom. Appl.*, **1** no. 1 (2020) 45–53.

[21] M. L. Zeinali, Flag curvature of invariant 3-power metrics on homogeneous spaces, *J. Finsler Geom. Appl.*, **4** no. 1 (2023) 124–132.

Dariush Latifi

Department of Mathematics, University of Mohaghegh Ardabili, P.O.Box 5619911367, Ardabil, Iran

Email: latifi@uma.ac.ir

Milad L. Zeinali

Department of Mathematics, University of Mohaghegh Ardabili, P.O.Box 5619911367, Ardabil, Iran

Email: miladzeinali@gmail.com

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