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BSE NORM FOR ABSTRACT SEGAL ALGEBRAS

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ABSTRACT. Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. In this paper, we first recall and study three important and practical mappings ${}_{\mathcal{A}}L$, Γ_1 and Γ_2. Then we investigate whenever these mappings have closed ranges. In fact, we research and study the conditions, under which having closed range of one of these mappings implies having the closed range of the another mapping. After that, using these results, we give a necessary and sufficient condition for $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, to be an algebra with BSE norm. Finally, we generalize some general results about abstract Segal algebras with respect to natural Banach functional algebras for abstract Segal algebras with respect to arbitrary Banach algebras. Also, throughout the paper, we provide examples to clarify the stated content.

1. Introduction

In 1990, Takahasi and Hatori introduced and studied the notion of BSE algebras [21]. Subsequently, several authors investigated this concept for various kinds of commutative Banach algebras; see for example [1], [2], [9], [10], [13] and [14], [21], [22] and [23]. Moreover, as the recent works and also some valuable survey works we refer to [6] and [7].

To provide the definition of BSE algebras, we present some preliminaries, as the following.

Keywords: Abstract Segal algebra, BSE algebra, BSE function, BSE norm, commutative Banach algebra, operator norm.

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Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative Banach algebra with the dual space $\mathcal{A}^*$. Denote by $\Delta(\mathcal{A})$, the linearly independent subspace of $\mathcal{A}^*$, consisting of all nonzero multiplicative linear functionals on $\mathcal{A}$, called the Gelfand (character) space of $\mathcal{A}$. Note that $\Delta(\mathcal{A})$ is always considered with the weak* topology, inherited from $\mathcal{A}^*$ and it is a locally compact Hausdorff space [12, Theorem 2.2.3]. Furthermore, we denote by $C_b(\Delta(\mathcal{A}))$, the Banach algebra consisting of all continuous and bounded complex valued functions on $\Delta(\mathcal{A})$, equipped with the pointwise product and supremum norm. Consider the Gelfand mapping of $\mathcal{A}$, defined as

$$\mathcal{A} \to C_b(\Delta(\mathcal{A})) \qquad a \mapsto \widehat{a},$$

where $\widehat{a}(\varphi) = \varphi(a)$ $(\varphi \in \Delta(\mathcal{A}))$ and set $\widehat{\mathcal{A}} = \{\widehat{a} : \ a \in \mathcal{A}\}$. Then $\mathcal{A}$ is called semisimple if its Gelfand mapping is injective. Throughout the paper, $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a commutative and semisimple Banach algebra.

A function algebra on a locally compact Hausdorff space X is in fact a subalgebra $\mathcal{A}$ of $C_b(X)$, separating the point of X; in the sense that the following two conditions are satisfied;

(i). For each $x \in X$, there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$.

(ii). Foe any $x, y \in X$ with $x \neq y$, there exists $g \in \mathcal{A}$ such that $g(x) \neq g(y)$.

Moreover, $\mathcal{A}$ is called a Banach function algebra, if there exists some norm $\|\cdot\|$ on $\mathcal{A}$ such that $(\mathcal{A}, \|\cdot\|)$ is a Banach algebra. In addition, $\mathcal{A}$ is called a natural Banach function algebra if

$$\Delta(\mathcal{A}) = \{\varepsilon_x : \ x \in X\},$$

where $\varepsilon_x(f) = f(x)$ $(f \in \mathcal{A})$.

Following [21], the function $\sigma \in C_b(\Delta(\mathcal{A}))$ is called a BSE function if there exists a constant $M > 0$ such that the inequality

$$\left|\sum_{j=1}^{n} d_j \sigma(\varphi_j)\right| \leq M \left\|\sum_{j=1}^{n} d_j \varphi_j\right\|_{\mathcal{A}^*}$$

holds, for every finite number of $\varphi_1, \ldots, \varphi_n$ in $\Delta(\mathcal{A})$ and the same number of complex numbers $d_1, \ldots, d_n$. The BSE norm of σ $(\|\sigma\|_{BSE,\mathcal{A}})$, is the infimum of all such M. The set of all BSE$-$functions is denoted by $C_{\mathrm{BSE}}(\Delta(\mathcal{A}))$. Takahasi and Hatori [21, Lemma 1] showed that $(C_{\mathrm{BSE}}(\Delta(\mathcal{A})), \|\cdot\|_{\mathrm{BSE},\mathcal{A}})$, is always a commutative and semisimple Banach algebra. Note that

$$\widehat{\mathcal{A}} \subseteq C_{\mathrm{BSE}}(\Delta(\mathcal{A}))$$

and

$$\|\widehat{x}\|_\infty \leq \|\widehat{x}\|_{\mathrm{BSE},\mathcal{A}} \leq \|x\| \qquad (x \in \mathcal{A}).$$

Following [7] and also [22], $\mathcal{A}$ is called a BSE norm algebra (or has a BSE norm) if there exists $K > 0$ such that $\|x\| \leq K \|\widehat{x}\|_{\mathrm{BSE},\mathcal{A}}$ $(x \in \mathcal{A})$.

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A bounded and linear operator T on $\mathcal{A}$ is called a multiplier if $xT(y) = T(x)y$ $(x, y \in \mathcal{A})$. It should be noted that by [12, Proposition 1.4.11], any multiplier T satisfies the following equality;

$$T(xy) = xT(y) = T(x)y \qquad (x, y \in \mathcal{A}).$$

Then the set $M(\mathcal{A})$, consisting of all multipliers of $\mathcal{A}$, is a unital commutative Banach algebra and it is called the multiplier algebra of $\mathcal{A}$. By [18, Theorem 1.2.2], for any $T \in M(\mathcal{A})$ there exists a unique function $\widehat{T} \in C_b(\Delta(\mathcal{A}))$ such that

$$\widehat{T(x)}(\varphi) = \widehat{T}(\varphi)\widehat{x}(\varphi),$$

for all $x \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$. Furthermore, $\|\widehat{T}\|_\infty \leq \|T\|$ $(T \in M(\mathcal{A}))$. Let

$$\widehat{M(\mathcal{A})} = \{\widehat{T} : T \in M(\mathcal{A})\}.$$

Then $\mathcal{A}$ is called a BSE algebra if

$$C_{\mathrm{BSE}}(\Delta(\mathcal{A})) = \widehat{M(\mathcal{A})}.$$

Following [11], a bounded net $\{x_\alpha\}_{\alpha\in\Lambda}$ in $\mathcal{A}$, is called a bounded $\Delta-$weak approximate identity for $\mathcal{A}$, if $\lim_\alpha \varphi(ax_\alpha) = \varphi(a)$, for any $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$ In [21, Corollary 5], the authors proved that $\mathcal{A}$ has a bounded $\Delta-$weak approximate identity if and only if

$$\widehat{M(\mathcal{A})} \subseteq C_{\mathrm{BSE}}(\Delta(\mathcal{A})).$$

It follows that all BSE algebras possess a bounded $\Delta-$weak approximate identity.

In this paper, since we are dealing with different commutative Banach algebras, to avoid ambiguity, we denote by ${}_{\mathcal{A}}L$, the natural mapping

$$L : \mathcal{A} \to M(\mathcal{A}) \qquad (a \mapsto L_a),$$

where L_a is the left multiplication operator at a, defined as $L_a(b) = ab$ $(b \in \mathcal{A})$. Moreover in stead of $C_b(\Delta(\mathcal{A}))$, we denote the range of the Gelfand mapping by $(C_0(\Delta(\mathcal{A})), \|\cdot\|_\infty)$ and $(C_{\mathrm{BSE}}(\Delta(\mathcal{A})), \|\cdot\|_{\mathrm{BSE}})$, respectively. We also indicate these mapping with Γ_1 and Γ_2, respectively. Then we investigate when these mapping have closed range. These results are applied in the main achievements of the paper. Afterwards, as a main result we show in the class of BSE algebras that any abstract Segal algebra $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, possessing a bounded $\Delta-$weak approximate identity, is a BSE norm algebra if and only if $\mathcal{B} = \mathcal{A}$. Finally, some general results about abstract Segal algebras of natural Banach function algebras, have been generalized for arbitrary abstract Segal algebras. Furthermore, some particular examples are provided for clarification.

2. Main Results

Lemma 2.1. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra with a bounded $\Delta-$weak approximate identity. Then there exists $\beta > 0$ such that for each $T \in M(\mathcal{A})$ we have*

$$\|\widehat{T}\|_{\mathrm{BSE}} \leq \beta \|T\|.$$

Lemma 2.2. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra. Then $\widehat{a} = \widehat{{}_{\mathcal{A}}L_a}$ and $\|\widehat{a}\|_{\mathrm{BSE},\mathcal{A}} = \|\widehat{{}_{\mathcal{A}}L_a}\|_{\mathrm{BSE}}$, for each $a \in \mathcal{A}$.*

Proposition 2.3. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra. Consider the following assertions.*

(i). Γ_1 *has closed range.*
(ii). Γ_2 *has closed range.*
(iii). ${}_{\mathcal{A}}L$ *has closed range.*

Then (i) $\Rightarrow$ (ii). Moreover, if $\mathcal{A}$ has a bounded $\Delta-$weak approximate identity then (ii) $\Rightarrow$ (iii).

Following [5], the Banach algebra $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is called an abstract Segal algebra with respect to $\mathcal{A}$ if the following conditions are satisfied.

(i) $\mathcal{B}$ is an essential dense ideal in $\mathcal{A}$; i.e.

$$\mathcal{B} = \{ab : \ a \in \mathcal{A}, b \in \mathcal{B}\}.$$

(ii) There exists $M > 0$ such that $\|b\|_{\mathcal{A}} \leq M\|b\|_{\mathcal{B}}$, for any $b \in \mathcal{B}$.
(iii) There exists $N > 0$ such that $\|ab\|_{\mathcal{B}} \leq N\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}$, for all $a, b \in \mathcal{B}$.

Since in the definition of abstract Segal algebras, the essentiality of $\mathcal{B}$ is assumed, it follows that $\mathcal{B}$ is semisimple. Moreover, $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$ are homeomorphic; see [5, Theorem 2.1]. In fact, $\widetilde{\varphi} = \varphi|_{\mathcal{B}}$ belongs to $\Delta(\mathcal{B})$, for any $\varphi \in \Delta(\mathcal{A})$ and

$$\Delta(\mathcal{B}) = \{\widetilde{\varphi} : \ \varphi \in \Delta(\mathcal{A})\};$$

see [3, Lemma 2.2]. For any $\sigma \in C_b(\Delta(\mathcal{A}))$, define $\widetilde{\sigma}$ on $\Delta(\mathcal{B})$ as

$$\widetilde{\sigma}(\widetilde{\varphi}) = \sigma(\varphi).$$

Lemma 2.4. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. Then*

$$C_b(\Delta(\mathcal{B})) = \{\widetilde{\sigma} : \ \sigma \in C_b(\Delta(\mathcal{A}))\}.$$

Moreover, $\|\widetilde{\sigma}\|_\infty = \|\sigma\|_\infty$, for any $\sigma \in C_b(\Delta(\mathcal{A}))$.

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For each $f \in \mathcal{A}^*$, the restriction of f to $\mathcal{B}$ is denoted by $\widetilde{f} = f|_{\mathcal{B}}$. Moreover, $\widetilde{f} \in \mathcal{B}^*$ and $\|\widetilde{f}\|_{\mathcal{B}^*} \leq M\|f\|_{\mathcal{A}^*}$. Now we have the next result.

Lemma 2.5. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. If $\widetilde{\sigma} \in C_{\mathrm{BSE}}(\Delta(\mathcal{B}))$ then $\sigma \in C_{\mathrm{BSE}}(\Delta(\mathcal{A}))$ and $\|\sigma\|_{\mathrm{BSE},\mathcal{A}} \leq M\|\widetilde{\sigma}\|_{\mathrm{BSE},\mathcal{B}}$.*

Corollary 2.6. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. Then for each $b \in \mathcal{B}$,*

$$\|\widehat{b}\|_{\mathrm{BSE},\mathcal{A}} \leq M\|\widehat{b}\|_{\mathrm{BSE},\mathcal{B}}.$$

Proposition 2.7. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. If ${}_{\mathcal{B}}L$ is bounded from below then $\mathcal{B} = \mathcal{A}$.*

By [7, Theorem page. 40], if the Banach algebra $\mathcal{A}$ is a BSE algebra, then $\mathcal{A}$ is a BSE norm algebra if and only if ${}_{\mathcal{A}}L$ is bounded from below. Now we have the next result, as a generalization of [7, page. 67], which has been presented for general Segal algebras.

Theorem 2.8. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a* BSE *algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$ with a bounded $\Delta-$weak approximate identity. If $\mathcal{B}$ is a* BSE *norm algebra then $\mathcal{B} = \mathcal{A}$.*

Proposition 2.9. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a* BSE *algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. Then the following assertions are equivalent.*

(i). *$\mathcal{B}$ is a* BSE *norm algebra with a bounded $\Delta-$weak approximate identity.*

(ii). *$\mathcal{B} = \mathcal{A}$ and $\mathcal{A}$ is a* BSE *norm algebra.*

We conclude this paper with the next result. It is in fact a generalization of [6, Theorem page. 34], which is regarding to natural Banach function algebras.

Theorem 2.10. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative and semisimple Banach algebra and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be an abstract Segal algebra with respect to $\mathcal{A}$. Suppose that $\mathcal{A}$ has a* BSE *norm and $\mathcal{B}$ possesses a bounded $\Delta-$weak approximate identity. Then three norms $\|\cdot\|_{\mathcal{A}}$, $\|\cdot\|_{\mathrm{BSE},\mathcal{A}}$ and $\|\cdot\|_{\mathrm{BSE},\mathcal{B}}$ are equivalent on $\mathcal{B}$.*

3. Conclusions

In this paper, we recall and study three mappings ${}_{\mathcal{A}}L$, Γ_1 and Γ_2. Then we investigate whenever these mappings have closed ranges. Using these results, we give a necessary and sufficient condition for $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, to be an algebra with BSE norm. Some general results about abstract Segal algebras with respect to natural Banach functional algebras are generalized for abstract Segal algebras with respect to arbitrary Banach algebras.

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