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COMPLETE CLASSIFICATION OF HOMOGENEOUS STRUCTURES ON LORENTZIAN DIRECT EXTENSIONS OF THE HEISENBERG GROUP

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Abstract. The Heisenberg Lie group is one of the most famous and important Lie groups among the family of three dimensional Lie groups. The direct extension of this group to the fourth dimension was taken into consideration in the study of the nilpotent Lie algebras from the fourth dimension, and as a result, the classification of these extensions up to isometric classes was previously presented in some research. Homogeneous structures provide us with a tensor approach to investigate the homogeneity of space. Perhaps the most important feature of homogeneous structures can be summarized in this statement that in Riemannian geometry, the existence of homogeneous structures is equivalent to being reductive locally homogeneous of space. In this paper, based on the existing classification of the direct Lorentzian extension of the Heisenberg group with dimension four, which are isometrically classified in the form of five families, we study the family of homogeneous structures on this space and classify them completely. In non-flat cases, we determine the homogeneous structures separately in each class.

1. Introduction

Homogeneous spaces are significant in theoretical physics and differential geometry studies. A (pseudo-) Riemannian manifold (M, g) is defined such that for any two points $p, q \in M$, an isometry ϕ exists on M with $\phi(p) = q$. In simpler terms, the isometry group $I(M)$ of M acts transitively on it. While Lie groups and symmetric spaces are basic examples of homogeneous spaces, they encompass a

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broader family and offer valuable instances in geometric investigations. Each homogeneous manifold M can be represented as the quotient manifold G/H of a Lie group G and a closed subgroup H. This representation, though not unique, provides an algebraic framework for studying homogeneous spaces and simplifies calculations. For instance, reference [20] employed an algebraic approach to examine the isometric geometry of symmetric spaces beyond four dimensions. Similarly, reference [2] utilized an algebraic approach to analyze the Walker structures and Ricci solitons on conformally flat homogeneous pseudo-Riemannian Lie groups in four dimensions.

By employing homogeneous structures, a tensorial approach can be adopted to analyze the presented homogenous spaces. Ambrose and Singer initially defined homogeneous structures in Riemannian mode [1], and Tericerri and Vanhecke subsequently studied them [18]. They were precisely positioned and subsequently expanded to the pseudo-Riemannian state by Gadea and Oubina [9]. We refer to the homogeneous space G/H as reductive whenever the Lie algebra $\mathfrak{g}$ of G can be decomposed as $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, where $\mathfrak{h}$ is a Lie algebra of H and $\mathfrak{m}$ is an $\mathrm{Ad}(H)$ invariant subspace such that $\mathrm{Ad}(h)(\mathfrak{m}) \subset \mathfrak{m}$. The condition $\mathrm{Ad}(h)(\mathfrak{m}) \subset \mathfrak{m}$ implies that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, and the converse is also true when H is connected. The categorization of smooth homogenous structures has captivated the interest of researchers in differential geometry, as exemplified in references [8, 12, 13]. We shall defer further elaboration on homogeneous structures to the ensuing section.

The three-dimensional Heisenberg group H_3 is a mathematical concept that describes a geometric phenomenon commonly encountered in our daily lives. By selecting a plane in our three-dimensional space, we essentially create a Heisenberg group. For instance, the page you are currently reading can be considered a plane that generates a Heisenberg group. Furthermore, the act of taking photographs places one within the context of Heisenberg Group Theory, as it involves the transfer and encoding of information along a line onto a page, thereby forming a Heisenberg Group. The Heisenberg group also holds significance in the development of mathematical models used to study one-dimensional quantum mechanical systems. For a comprehensive understanding of the Heisenberg group and its applications in physics, please refer to the reference provided [3].

The Heisenberg group, is a three-dimensional Lie group that consists of real matrices of the form:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

It is homeomorphic to the Lie group $\mathbb{R}^3$ through the following multiplication operation:

$$(x, y, z)(\tilde{x}, \tilde{y}, \tilde{z}) = (x + \tilde{x}, y + \tilde{y}, z + \tilde{z} - x\tilde{y}).$$

The Heisenberg group holds a significant position in the classification of Riemannian and pseudo-Riemannian Lie groups, showcasing unique properties among three-dimensional Lie groups ([15, 17]).

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In the realm of physics, the concept of space-time emerges as a mathematical model that unifies three spatial dimensions and one temporal dimension into a four-dimensional framework. This model, known as a Lorentzian manifold in differential geometry, serves as a valuable tool for visualizing and comprehending relativistic effects, particularly the diverse perceptions of events by different observers in terms of location and timing.

Until the early 20th century, it was widely held that the three-dimensional geometry of the world, encompassing locations, shapes, distances, and directions, existed separately from time, which measured the occurrence of events. However, the introduction of the Lorentz transformation and the theory of special relativity revolutionized our understanding of space and time. In 1908, Hermann Minkowski presented a groundbreaking geometrical interpretation, essentially a mathematical model, that merged space and time into a four-dimensional framework. This framework, known as Minkowski space, became the cornerstone of the theory of special relativity. Minkowski's interpretation proved pivotal in the development of the theory of general relativity, which posits that mass and energy induce curvature in space-time. For further insights into Lorentzian manifolds and the broader concept of pseudo-Riemannian manifolds, please refer to the reference [16].

Let (M, g) be a connected (pseudo-)Riemannian manifold. Let ∇ and R denote the Levi-Civita connection and the curvature tensor of (M, g), respectively. A homogeneous (pseudo-)Riemannian structure on (M, g) is a tensor field T of type $(1, 2)$ on M such that the connection $\tilde{\nabla} = \nabla - T$ satisfies the following relationships:

$$\widetilde{\nabla} g = 0, \quad \widetilde{\nabla} R = 0, \quad \widetilde{\nabla} T = 0. \tag{1.1}$$

The following theorem clearly states the relationship between the presence of homogeneous structures on a manifold and its homogeneity feature:

Theorem 1.1. [9] *Let (M, g) be a connected (pseudo-)Riemannian manifold, simply connected and complete. In this case, (M, g) admits a homogeneous (pseudo-)Riemannian structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold.*

Certainly, a homogeneous structure T of this nature establishes a delivered decomposition of an appropriate coset description of (M, g), and vice versa. It is crucial to recognize that distinct homogeneous structures on (M, g) can result in different representations of M as a coset space. A comprehensive discussion of homogeneous structures and recent research findings in this field can be found in the reference [6].

When T represents a homogeneous structure on (M, g), we utilize T to denote both the tensor field of order $(1, 2)$ and its metric equivalent tensor field of order $(0, 3)$, which is defined by $T(X, Y, Z) = g(T_X Y, Z)$. We will employ this notation throughout our analysis.

Let's consider a fixed point x in M and an orthogonal base for the tangent space T_xM. Let's also consider the vector space $V = \mathbb{R}^m$ equipped with the symmetric bilinear form $\langle\ ,\ \rangle$ of the signature (p, q) as a model for the space (T_xM, g_x). Now, let's consider the tensor space $\mathcal{S}(V) \subset \otimes^3 V^*$ whose members have the usual symmetries of homogeneous structures (symmetries arising from the condition $\widetilde{\nabla} g = 0$). In fact,

$$\mathcal{S}(V) = \{T \in \otimes^3 V | T(X, Y, Z) + T(X, Z, Y) = 0\}.$$

Given the condition that $\dim(M) \geq 3$, the space $\mathcal{S}(V)$ can be decomposed into $O(p, q)$-indecomposable and two-by-two orthogonal submodules as follows:

$$\mathcal{S}(V) = \mathcal{S}_1(V) \oplus \mathcal{S}_2(V) \oplus \mathcal{S}_3(V),$$

where

$$\mathcal{S}_1 = \Big\{T \in \mathcal{S} \,/\, T(X, Y, Z) = g(X, Y)\varphi(Z) - g(X, Z)\varphi(Y),\ \varphi \in \Omega^1(M)\Big\},$$

$$\mathcal{S}_2 = \Big\{T \in \mathcal{S} \,/\, \sigma_{X,Y,Z} T(X, Y, Z) = 0,\ c_{12}(T) := \sum_{i=1}^{n} \varepsilon_i T(e_i, e_i, \cdot) = 0\Big\},$$

$$\mathcal{S}_3 = \Big\{T \in \mathcal{S} \,/\, T(X, Y, Z) + T(Y, X, Z) = 0\Big\}.$$

In addition, $\sigma_{X,Y,Z}$ denotes the cyclic summation to X, Y, Z. Homogeneous structures that belong to one of the aforementioned submodules or to the direct sum of two of them possess specific meanings and properties, notably:

- $T \in \mathcal{S}_3$ if and only if

(1.2) $$g([X, Y]_{\mathfrak{m}}, Z) + g([X, Z]_{\mathfrak{m}}, Y) = 0, \quad \forall X, Y, Z \in \mathfrak{m}.$$

  Naturally reductive homogeneous manifold $(M = G/H, g)$ is defined with this property. In this case, the geodesics corresponding to the Levi-Civita connection (M, g) and the canonical connection of the delivery decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ are the same.
- $T \in \mathcal{S}_1 \oplus \mathcal{S}_2$ if and only if $\sigma_{X,Y,Z} T(X, Y, Z) = 0$. In this case, we call (M, g) cyclic homogeneous. In the Riemannian case, the cyclic Lie groups and the cyclic homogeneous spaces were studied in [10] and [11] references, respectively. However, the cyclic Lie groups from the fourth dimension were investigated with Lorentzian sign in [7] and with neutral sign in [19].
- The projection of the homogeneous structure T on the subspaces $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are defined by the following relations, respectively

$$p_1(T)(x, y, z) = \frac{1}{2}\langle x, y\rangle c_{12}(T)(z) - \frac{1}{2}\langle x, z\rangle c_{12}(T)(y),$$

$$p_2(T)(x, y, z) = (T - p_1(T) - p_3(T))(x, y, z),$$

$$p_3(T)(x, y, z) = \frac{1}{3}\sigma_{x,y,z} T(x, y, z).$$

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In the realm of Lorentzian groups, Heisenberg's contributions gained prominence within the context of nilpotent algebra classification in four dimensions. Magnin, as cited in reference [14], demonstrated that up to isomorphism, only two non-Abelian Lie algebras exist in dimension four. These are denoted as $\mathfrak{h}_3 \oplus \mathfrak{r}$ and $\mathfrak{g}_4$, corresponding to the Lie groups $H_3 \times \mathbb{R}$ and G_4, respectively. Subsequently, reference [4] presented a classification of left invariant Lorentzian metrics up to isometries on these Lie groups. The ensuing fundamental theorem serves as a cornerstone for this classification.

Theorem 1.2. [4] *Every left invariant Lorentzian metric on the direct extension of the Heisenberg group $H_3 \times \mathbb{R}$ up to automorphisms of $H_3 \times \mathbb{R}$ with respect to the basis $\{e_1, \cdots, e_4\}$ is isometric with one of the metrics below*

$$g_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, 0 < \mu \in \mathbb{R}, \quad g_\lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \varepsilon\lambda & 0 \\ 0 & 0 & 0 & -\varepsilon \end{pmatrix}, 0 < \lambda \in \mathbb{R}, \varepsilon^2 = 1,$$

$$g_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.3}$$

$$g_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

This theorem provides a suitable basis for the study of homogeneous structures. Here, it is important to pay attention to the point that the Lie algebra corresponding to the direct extension of the Heisenberg group, i.e. $\mathfrak{h}_3 \times \mathfrak{r}$ in the above basis is $[e_1, e_2] = e_3$. It is necessary to check the equations related to homogeneous structures and determine the available solutions. We will discuss this issue in the next section.

2. Main Results

In this section, we examine the equations related to homogeneous structures on different classes of direct extensions of the Heisenberg group $H_3 \times \mathbb{R}$ from the Lorentz signature and fully classify the homogeneous structures in each family.

Class $(H_3 \times \mathbb{R}, g_\mu)$**:** Consider the direct extension $H_3 \times \mathbb{R}$ equipped with the Lorentzian metric g_μ introduced in the relation (1.3). If we set $\Lambda_i := \nabla_{e_i}$, using the famous Koszul formula, the Levi-Civita connection components are obtained as follows

$$
\begin{aligned}
&\Lambda_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & 0 & \frac{\mu}{2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
&\Lambda_3 = \begin{pmatrix} 0 & \frac{\mu}{2} & 0 & 0 \\ \frac{\mu}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_4 = 0.
\end{aligned} \tag{2.1}
$$

Now, by using the relation $R_{ij} := R(e_i, e_j) = [\nabla_{e_i}, \nabla_{e_j}] - \nabla_{[e_i, e_j]}$, non-zero components of the curvature tensor are calculated as

$$
\begin{aligned}
&R_{12} = \begin{pmatrix} 0 & -\frac{3}{4}\mu & 0 & 0 \\ -\frac{3}{4}\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_{13} = \begin{pmatrix} 0 & 0 & -\frac{1}{4}\mu^2 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{4}\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
&R_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4}\mu^2 & 0 \\ 0 & -\frac{1}{4}\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{2.2}
$$

The condition $\nabla R = 0$ yields the unacceptable solution $\mu = 0$, so this family is never symmetric. Now suppose that T is a homogeneous structure with their usual symmetries on $(H_3 \times \mathbb{R}, g_\mu)$. The components of $\widetilde{\nabla} = \nabla - T$ can be calculated as follows

$$
\begin{array}{lll}
\widetilde{\nabla}_{112} = -T_{112}, & \widetilde{\nabla}_{113} = -T_{113} & \widetilde{\nabla}_{114} = -T_{114}, \\
\widetilde{\nabla}_{123} = -T_{123} + \frac{\mu}{2}, & \widetilde{\nabla}_{124} = -T_{124}, & \widetilde{\nabla}_{134} = -T_{134}, \\
\widetilde{\nabla}_{212} = -T_{212}, & \widetilde{\nabla}_{213} = -T_{213} - \frac{\mu}{2} & \widetilde{\nabla}_{214} = -T_{214}, \\
\widetilde{\nabla}_{223} = -T_{223}, & \widetilde{\nabla}_{224} = -T_{224}, & \widetilde{\nabla}_{234} = -T_{234}, \\
\widetilde{\nabla}_{312} = -T_{312} - \frac{\mu}{2}, & \widetilde{\nabla}_{313} = -T_{313} & \widetilde{\nabla}_{314} = -T_{314}, \\
\widetilde{\nabla}_{323} = -T_{323}, & \widetilde{\nabla}_{324} = -T_{324}, & \widetilde{\nabla}_{334} = -T_{334}, \\
\widetilde{\nabla}_{412} = -T_{412}, & \widetilde{\nabla}_{413} = -T_{413} & \widetilde{\nabla}_{414} = -T_{414}, \\
\widetilde{\nabla}_{423} = -T_{423}, & \widetilde{\nabla}_{424} = -T_{424}, & \widetilde{\nabla}_{434} = -T_{434}.
\end{array} \tag{2.3}
$$

According to the symmetries of T, clearly $\widetilde{\nabla} g = 0$. If we put $\mathfrak{R}_{ijkl;r} = (\widetilde{\nabla}_{e_r} R)_{ijkl}$, by performing standard calculations, the non-zero components of the derivative of the curvature tensor with respect to the connection $\widetilde{\nabla}$ are as below

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$$\begin{aligned}
&\mathfrak{R}_{1213;1} = -\tfrac{1}{2}\mu(2T_{123} - \mu), && \mathfrak{R}_{1213;r} = -T_{r23}\mu, r = 2, 3, 4,\\
&\mathfrak{R}_{1214;r} = -\tfrac{3}{4}\mu T_{r24}, r = 1, \cdots, 4, && \mathfrak{R}_{1223;2} = -\tfrac{1}{2}\mu(2T_{213} + \mu),\\
&\mathfrak{R}_{1223;r} = -T_{r13}\mu, r = 1, 3, 4, && \mathfrak{R}_{1224;r} = -\tfrac{3}{4}\mu T_{r14}, r = 1, \cdots, 4,\\
&\mathfrak{R}_{1314;r} = \tfrac{1}{4}\mu T_{r34}, r = 1, \cdots, 4, && \mathfrak{R}_{1334;r} = -\tfrac{1}{4}\mu^2 T_{r14}, r = 1, \cdots, 4,\\
&\mathfrak{R}_{2324;r} = -\tfrac{1}{4}\mu T_{r34}, r = 1, \cdots, 4 && \mathfrak{R}_{2334;r} = -\tfrac{1}{4}\mu^2 T_{r24}, r = 1, \cdots, 4.
\end{aligned}$$

Therefore, according to the condition $\widetilde{\nabla} R = 0$ and $\mu > 0$, it gives

$$\begin{aligned}
&T_{123} = \tfrac{\mu}{2}, T_{213} = -\tfrac{\mu}{2}, T_{223} = T_{323} = T_{423} = T_{113} = T_{313} = T_{413} = 0,\\
&T_{r14} = T_{r24} = T_{r34} = 0, r = 1, \cdots, 4.
\end{aligned}$$

Now we proceed to the calculation of $\widetilde{\nabla} T$. If we put $\mathfrak{T}_{ijk;l} = (\widetilde{\nabla}_{e_l} T)_{ijk}$ have

$$\mathfrak{T}_{212;1} = -T_{112}^2, \mathfrak{T}_{112;2} = -T_{212}^2,$$

which immediately results $T_{112} = T_{212} = 0$, according to the relation $\widetilde{\nabla} T = 0$. In this case, $\widetilde{\nabla} T$ becomes zero by itself, and therefore if we put $T_{312} = \kappa$ and $T_{412} = \eta$, the homogeneous structure of T will be obtained as follows

$$T = \mu(e^1 \otimes e^2 \wedge e^3 - e^2 \otimes e^1 \wedge e^3) + 2\kappa(e^3 \otimes e^1 \wedge e^2) + 2\eta(e^4 \otimes e^1 \wedge e^2), \kappa, \eta \in \mathbb{R}.$$

The projection of this homogeneous structure on the subspaces $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ is as follows

$$\begin{aligned}
p1(T) &= 0,\\
p2(T) &= \tfrac{1}{3}(\mu - 2\kappa)(e^1 \otimes e^2 \wedge e^3 - e^2 \otimes e^1 \wedge e^3) - \tfrac{2}{3}\eta(e^1 \otimes e^2 \wedge e^4 - e^2 \otimes e^1 \wedge e^4)\\
&\quad -\tfrac{2}{3}(\mu - 2\kappa)e^3 \otimes e^1 \wedge e^2 + \tfrac{4}{3}\eta e^4 \otimes e^1 \wedge e^2,\\
p3(T) &= \tfrac{2}{3}(\mu + \kappa)(e^1 \otimes e^2 \wedge e^3 - e^2 \otimes e^1 \wedge e^3 + e^3 \otimes e^1 \wedge e^2)\\
&\quad +\tfrac{2}{3}\eta(e^1 \otimes e^2 \wedge e^4 - e^2 \otimes e^1 \wedge e^4 + e^4 \otimes e^1 \wedge e^2).
\end{aligned}$$

According to the previous arguments, we state the following theorem.

Theorem 2.1. *Let $(H_3 \times \mathbb{R}, g_\mu)$ be a direct extension of the Heisenberg group with the metric g_μ introduced in the relation (1.3). In this case, all homogeneous structures $(H_3 \times \mathbb{R}, g_\mu)$ are as follows:*

$$T = \mu(e^1 \otimes e^2 \wedge e^3 - e^2 \otimes e^1 \wedge e^3) + 2\kappa(e^3 \otimes e^1 \wedge e^2) + 2\eta(e^4 \otimes e^1 \wedge e^2), \kappa, \eta \in \mathbb{R},$$

where κ and μ are arbitrary real coefficients. Also, this homogeneous structure belongs to the class $\mathcal{S}_3$ when $\mu = \eta - 2\kappa = 0$ and belongs to the class $\mathcal{S}_2$ when $\eta = \mu + \kappa = 0$. Otherwise, it is of the type $\mathcal{S}_2 \oplus \mathcal{S}_3$. Also, there is no canonical homogeneous structure (i.e., $\widetilde{\widetilde{\nabla}} = 0$).

In accordance with similar arguments, the subsequent theorems present a comprehensive categorization of homogeneous structures throughout the direct extensions of the Heisenberg group.

Theorem 2.2. *Let $(H_3 \times \mathbb{R}, g_\lambda)$ be a direct extension of the Heisenberg group with the metric g_λ introduced in the relation (1.3). In this case, all homogeneous structures $(H_3 \times \mathbb{R}, g_\lambda)$ are as follows*

$$T = \varepsilon\lambda(e^1 \otimes e^2 \wedge e^3 - e^2 \otimes e^1 \wedge e^3) + 2\kappa(e^3 \otimes e^1 \wedge e^2) + 2\eta(e^4 \otimes e^1 \wedge e^2), \kappa, \eta \in \mathbb{R}.$$

where κ, η are arbitrary real coefficients. Also, this homogeneous structure belongs to the class $\mathcal{S}_3$ when $\eta = \varepsilon\lambda - 2\kappa = 0$ and belongs to the class $\mathcal{S}_2$ when $\eta = \varepsilon\lambda + \kappa = 0$ and otherwise it is of the type $\mathcal{S}_2 \oplus \mathcal{S}_3$. Also, there is no canonical homogeneous structure (i.e., $\widetilde{\nabla} = 0$).

Theorem 2.3. *Let $(H_3 \times \mathbb{R}, g_1)$ be a direct extension of the Heisenberg group with the metric g_1 introduced in the relation (1.3). In this case, all homogeneous structures $(H_3 \times \mathbb{R}, g_1)$ are in one of the following families*

(i)

$$T = 2\kappa(e^1 \otimes e^2 \wedge e^4 - e^2 \otimes e^1 \wedge e^4 + e^4 \otimes e^1 \wedge e^2), \quad 0 \neq \kappa \in \mathbb{R}.$$

This family of homogeneous structures always belongs to the class $\mathcal{S}_3$.

(ii)

$$T = -(e^1 \otimes e^2 \wedge e^4 + e^2 \otimes e^1 \wedge e^4 - e^4 \otimes e^1 \wedge e^2) + 2\kappa e^4 \otimes e^1 \wedge e^4 + 2\eta e^4 \otimes e^2 \wedge e^4, \kappa, \eta \in \mathbb{R}.$$

This family of homogeneous structures belongs to the class $\mathcal{S}_3$ when $\kappa = \eta = 0$ and otherwise they are members of the class $\mathcal{S}_2 \oplus \mathcal{S}_3$.

(iii)

$$T = (e^1 \otimes e^2 \wedge e^4 - e^2 \otimes e^1 \wedge e^4) + 2\kappa e^3 \otimes e^1 \wedge e^2 + 2\eta e^4 \otimes e^1 \wedge e^2, \quad \kappa, \eta \in \mathbb{R}.$$

This family of homogeneous structures belongs to the class $\mathcal{S}_2$ when $\kappa = \eta + 1 = 0$ and belongs to the class $\mathcal{S}_3$ when $\kappa = 1 - 2\eta = 0$ and Otherwise, they are members of the class $\mathcal{S}_2 \oplus \mathcal{S}_3$. Also, this homogeneous structure is canonical whenever $\kappa = 1 + 2\eta = 0$.

Suppose we equip the direct extension $H_3 \times \mathbb{R}$ to the metric g_2 in the relation (1.3). In this case, the components of Levi-Civita connection will be as follows

$$\Lambda_1 = \Lambda_3 = \Lambda_4 = 0, \quad \Lambda_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.4}$$

Here, with straightforward calculations, it can be shown that $R \equiv 0$ and therefore this family of direct extensions of the Heisenberg group is flat. Therefore, the only remaining equation of the relation (1.1) for homogeneous structures is the relation $\widetilde{\nabla} T = 0$. The family of homogeneous structures for flat spaces is relatively large. In fact, a flat tunnel is isometric with $\mathbb{R}^n$ and its standard metric, therefore, in this study, we skip this obvious state (flat state) and study the next family.

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Theorem 2.4. *Let $(H_3 \times \mathbb{R}, g_3)$ be a direct extension of the Heisenberg group with the metric g_3 introduced in the relation (1.3). In this case, all homogeneous structures $(H_3 \times \mathbb{R}, g_3)$ are in one of the following families*

(i)

$$T = (e^1 \otimes e^2 \wedge e^3 - e^2 \otimes e^1 \wedge e^3) + 2\kappa e^2 \otimes e^1 \wedge e^2 + 2\eta e^3 \otimes e^1 \wedge e^2, \quad \kappa, \eta \in \mathbb{R}.$$

This family of homogeneous structures belongs to the class $\mathcal{S}_3$ when $\kappa = \eta - \frac{1}{2} = 0$ and also belongs to the class $\mathcal{S}_2$ when $\eta = -1$ and otherwise The member faces of the class $\mathcal{S}_2 \oplus \mathcal{S}_3$ are Also, canonical homogeneous structures arise whenever $\kappa = \eta + \frac{1}{2} = 0$.

(ii)

$$T = e^1 \otimes e^2 \wedge e^3 + 2\kappa e^2 \otimes e^1 \wedge e^2 - e^2 \otimes e^1 \wedge e^3 + 2\eta e^2 \otimes e^2 \wedge e^3 + e^3 \otimes e^1 \wedge e^2, \quad \eta, \kappa \in \mathbb{R}.$$

This family of homogeneous structures belongs to the class $\mathcal{S}_3$ when $\kappa = \eta = 0$ and otherwise they are members of the class $\mathcal{S}_2 \oplus \mathcal{S}_3$.

3. Conclusions

In this paper, we examined the direct extensions of the three-dimensional Heisenberg group, namely $H_3 \times \mathbb{R}$. We equipped the aforementioned extension, which is naturally four-dimensional, with the Lorentzian metric. According to prior research, this family of extensions of the Heisenberg group up to isometric classes falls within one of the five families specified in the 1.3 theorem. Therefore, we employed a computational approach to study each family of Lorentzian metrics individually and verified the equations pertaining to homogeneous structures. We systematically solved these equations and identified all feasible solutions. The outcomes of this classification of homogeneous structures were articulated in theorems 2.1, 2.2, 2.3, and 2.4. It is noteworthy that during the investigation, we discovered that the direct extensions of the Heisenberg group with the metric class g_2 are flat.

As mentioned in reference [8], Proposition 2-2, the Lorentzian simply connected Lie groups are equivalent to the Riemannian simply connected Lie groups of the same dimension. Therefore, this research focuses on investigating homogeneous structures on the special case of direct extension in the class family ii2 of the classification presented in Proposition 2-1 of the aforementioned reference. Future studies may explore homogeneous structures on other classes of Lorentzian Lie groups of the fourth dimension.

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