Mathematics and Society
University of Isfahan
https://math-sci.ui.ac.ir

Journal of Mathematics and Society
ISSN (print): 2345-6493, ISSN (on-line): 2345-6507
Vol. 9 No. 4 (2024), pp. 23-43.
© 2024 The Author(s)
https://www.ui.ac.ir

MODELING RAINFALL AND GROUNDWATER LEVEL DATA USING TIME-VARYING COPULA MODELS

HOSSEIN ZAMANII*, ZOHREH PAKDAMANI, MARZIEH SHEKARII

Abstract. Non-stationary data are often created when the observations of a study are collected sequentially in a time-dependent structure. In such a case, there will usually be a time trend with abrupt changes in the average or/and variance of the observations, which indicates that the data is non-stationary. To describe such data using statistical distributions and fitting parameters, time-varying models are suitable. The aim of this study is to introduce and apply time-varying models in which parameters are considered as time-varying in both marginal distributions and copula models. According to the monthly collection of rainfall and groundwater level data, the nature of these data is time-varying and the trend changes in these data shows that the average of data changes abruptly over time. To describe the correlation structure between these data, marginal distributions and then time-varying copulas have been used, so that the parameter of these models is considered to be vary over time as a function of time in the form of polynomial or exponential regression functions.

1. Introduction

Data stationary is an essential prerequisite in frequency analysis. But, in practice, in various field specially in economic due to collecting data sequentially in a reqular time or in hydrology, due to

Keywords: Time-varying Copula , Copula, Rainfall.
Article Type: Resaerch Paper.
Communicated by Khalil Shafie.
*Corresponding author.
Received: 01-01-2024, Accepted: 19-06-2024, Published Online: 20-07-2024.
Cite this article: H. Zamani, Z. Pakdaman and M. Shekari, Modeling Rainfall and Groundwater level data Using Time-varying copula models, *Journal of Mathematics and Society*, **9** no. 4 (2024) 23–43.
https://dx.doi.org/10.22108/msci.2024.140292.1637 .

climate changes, droughts, geological features, or human activities data often are non-stationarity in time series [13]. Any statistical frequency analysis based on the assumption of stationarity will be invalid if non-stationarity is not investigated in the hydrological time series[7]. Groundwater level is one of the hydrological variables that has suffered a significant drop due to over-extraction and decreased rainfall in most parts of the world [17]. Therefore, frequency analysis of this non-stationary series seems necessary.

The copula theory with its multivariate nature, provides great flexibility for modeling bivariate or multivariate data. In recent years, the use of copula functions in multivariate hydrological analysis has increased significantly. Copula functions have been widely used by different researchers for multivariate analysis of hydrological, hydrogeological and hydrometeorological data in recent years. for instance, several researchers [1, 2, 3, 5, 8, 12, 14, 18, 19] performed their multivariate analysis based on copula functions although the data were time dependent with time varying structure. Groundwater is one of the sources of fresh water in arid and semi-arid regions, which provides resilience against the lack of rainfall. But in the past few decades, due to population growth and agricultural and industrial development, there is an irreparable pressure on it which has caused a drop in the underground water level in most parts of the world, especially in Iran. Rainfall is one of the variables whose changes are effective on groundwater fluctuations, but the effects of increasing or decreasing rainfall on the groundwater level are always delayed due to several factors involving aquifer characteristics. Therefore, the aim of the current research is to investigate how the groundwater level has changed and most importantly, how the rainfall changes have been effective on the correlation structure of the rainfall with the groundwater changes.

2. **Main Results**

In statistical literature, the maximum likelihood method is a well-known approach for estimation the parameters of the models. To perform this, it is assumed that the observations are independent and identically distributed with a distribution that contains one or several parameters that are taken to be constant [**?**]. In some situations, however, the data are collected sequentially in a time dependent manner. In this case, a serial dependence is observed among the data which leads to a non-stationary distribution with non-constant parameters. Copula theory provides a theoretical basis for constructing multivariate models [6, 15]. The copula applies the marginal distribution functions to construct the joint distribution of random variables in order to describe the correlation between the random variables, and reflect the dependence structure. Suppose $Y_1^t, Y_2^t, \ldots, Y_n^t$ is a sequence of time dependent random variables that follows a distribution function $F^t = F(., \Theta^t)$ with a non-stationary mean and a non-constant variance. In the other word, $\Theta^t = (\theta_1^t, \ldots, \theta_k^t)$ contains one or

https://dx.doi.org/10.22108/msci.2024.140292.1637

more non-constant parameters which is a function of time $t\,(t = 1, 2, \ldots, n)$. Various researches considered different approach to handle the time-varying parameters in the model.For instance the CAS [1, 20], SCAR [20, 14],ARIMA [16], DCC [10] methods were used by authors for handling time-varying parameters.In this study, we incorporated the time as explanatory variable in the model through two polynomial regression models as below,

$$\theta_j^t = \varphi_j(t) = \beta_0 + \sum_{k=1}^{q} \beta_{jk} t^k \tag{2.1}$$

or

$$\theta_j^t = \phi_j(t) = \exp(\varphi_j(t)) \tag{2.2}$$

depends on whether there is a constrains on parameters to be positive or not. In equation (2.1) and (2.2), $\varphi_j(.)$ and $\phi_j(.)$ are the link functions and $\beta_j = (\beta_0, \beta_{j1}, \ldots, \beta_{jq})^{\mathrm{T}}$ are the regression parameters. In order to detect whether there is a variation in the mean and variance of the data through the time and select the degree of polynomial regression in the marginal model, the time series plot of the observation is investigated. In this study we applied Generalized gamma, Weibull, Log-normal and Gumbel probability distributions [9] for modeling the rainfall and groundwater series. These probability distributions are widely used in many studies specially in the hydrological studies. In this paper we applied special functional forms of the abovementioned models which was developed by Rigby and Stasinopoulos [19] as a generalized regression of univariate models called Generalized Addictive Models in Location, Scale and Shape (GAMLSS) with parameters $\pi = (\mu, \sigma, \upsilon)^{\mathrm{T}}$. Herein, the parameters μ, σ, υ are the location, scale, and shape parameters respectively. Based on the definition of the copula [18, 8], the time-varying copula can be expressed as

$$H_{Y_1^t, Y_2^t}(y_1^t, y_2^t) = C\left[F_1(y_1^t|\Theta_1^t), F_2(y_2^t|\Theta_2^t)|\xi^t\right] = C(u_1^t, u_2^t|\xi^t) \tag{2.3}$$

Where $C(.)$ represents the copula function and $F_j(.|\Theta_k^t) = u_j^t$, $(j = 1, 2)$ represents the time dependent marginal cumulative distribution functions and ξ^t is the time-varying copula parameters. Generally, a simple liner model $\xi^t = \delta_0 + \delta_1 t$, or its exponential form $\xi^t = \exp(\delta_0 + \delta_1 t)$ is suitable to describe the time-varying copula models.

3. Summary of Proofs

To investigate the proposed method practically, we conducted this study on the rainfall and groundwater level data which was collected during 1990 to 2023 from the Shamil basin in Hormozgan province. Figure 1 represents the time series plot of the rainfall and groundwater data during the study period. This figure shows that the rainfall series data changes in a quadratic form with a relatively high fluctuation around the mean. Therefore, it can be concluded that the mean of the

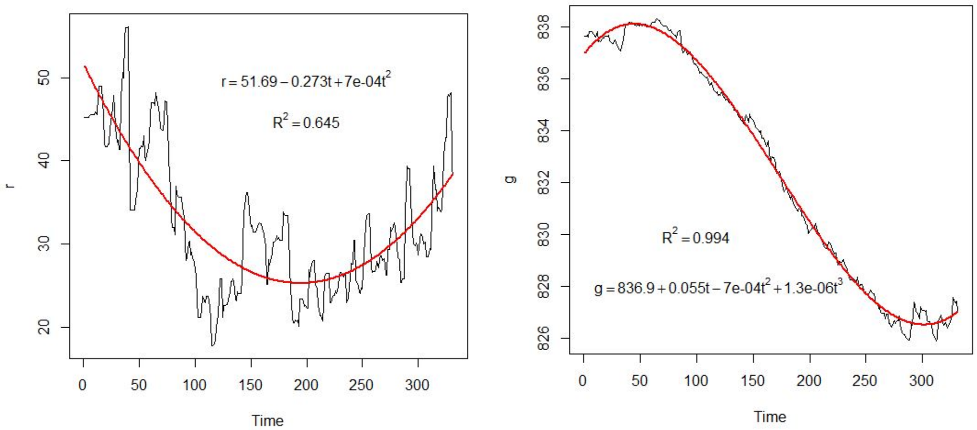

FIGURE 1. Rainfall (left) and Groundwater (right)

rainfall is a quadratic function of the time and its non-constant variance that can be considered as a linear function of the time. Moreover, as figure shows, the groundwater level changes in a cubic trend form whereas no fluctuation around the mean is observed. So that, we consider the mean of the groundwater to be a cubic function of the time with a constant dispersion parameter. Therefore, in both cases the time-varying distributions are distinguished to be appropriate models for fitting rainfall and groundwater series and obtaining the marginal cumulative distribution functions. Results of fitting marginal distributions to the rainfall and groundwater series with constant and time-varying parameters indicate that considering time varying parameters using appropriate polynomial regression according to Figure 1, significantly improves the results of AIC criterion. For instance, in fitting groundwater level using the fixed parameters Log-normal model the AIC=1927.44 while using the time-varying Log-normal model the AIC=187.96 which a reduction of 1739.48 has been achieved in the AIC.

The diagnostic plots of fitting time-varying Log-normal including the Q-Q plot and the Worm plot are illustrated in Figure 2. Both plots confirm the suitability of fitting time-varying Log-normal on rainfall and groundwater level series.

Figure 3 dsiplays fitting the fix parameter and the time-varying parameter copulas to the rainfall and groundwater data. As figure shows, the plot of time-varying copulas changes over time while the plot of fix copulas are constant. Generally, results of the study indicated that applying time-varying models, for both of marginal and copula functions, significantly decreas the AIC of the models. Therefore, this approach is useful when there exists autocorrelation among the data and the presumption of the MLE estimation is violated.

https://dx.doi.org/10.22108/msci.2024.140292.1637

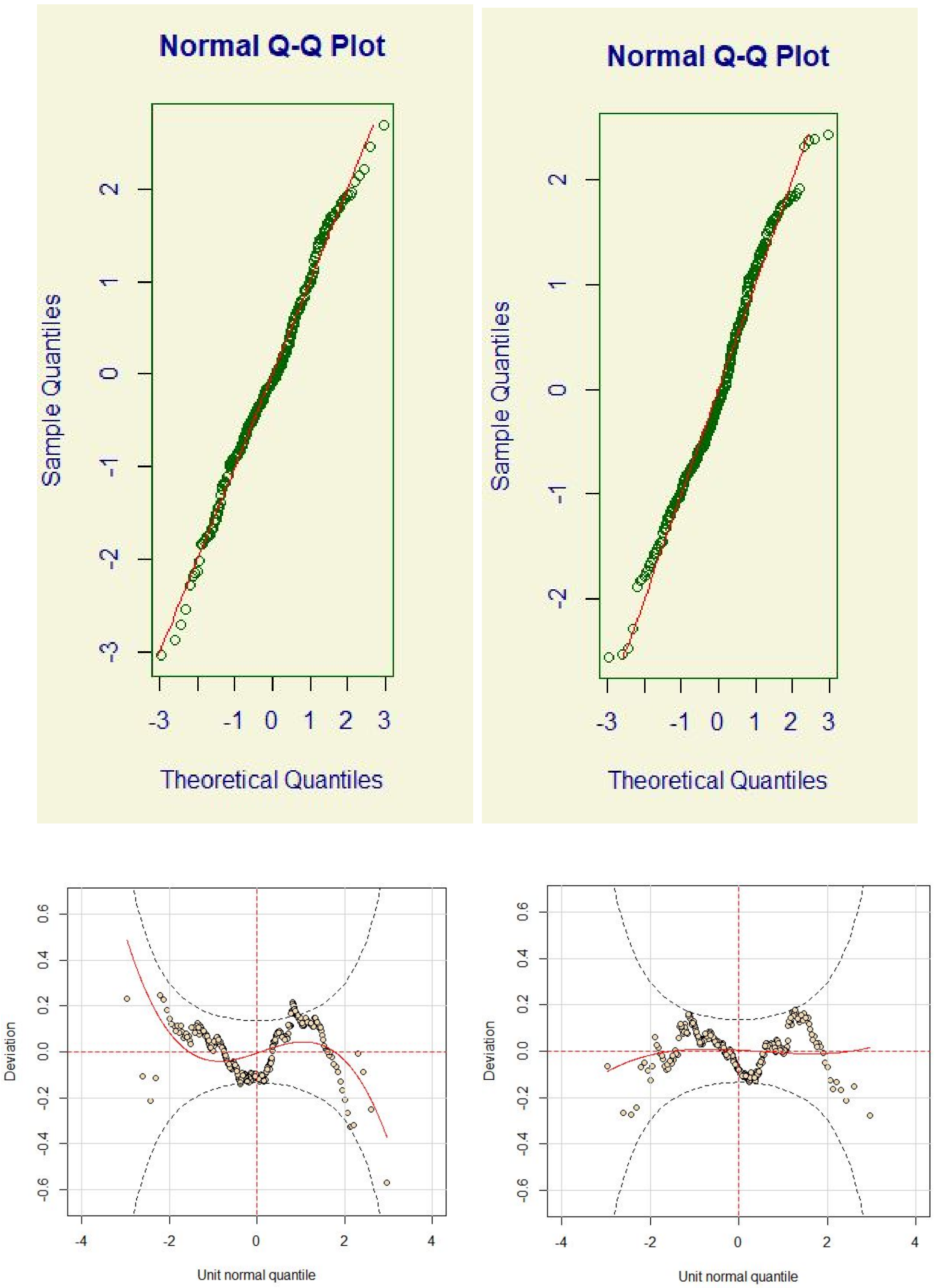

FIGURE 2. Q-Q plot and worm plot of fitting time-varying lognormal to Rainfall (left) and Groundwater (right)

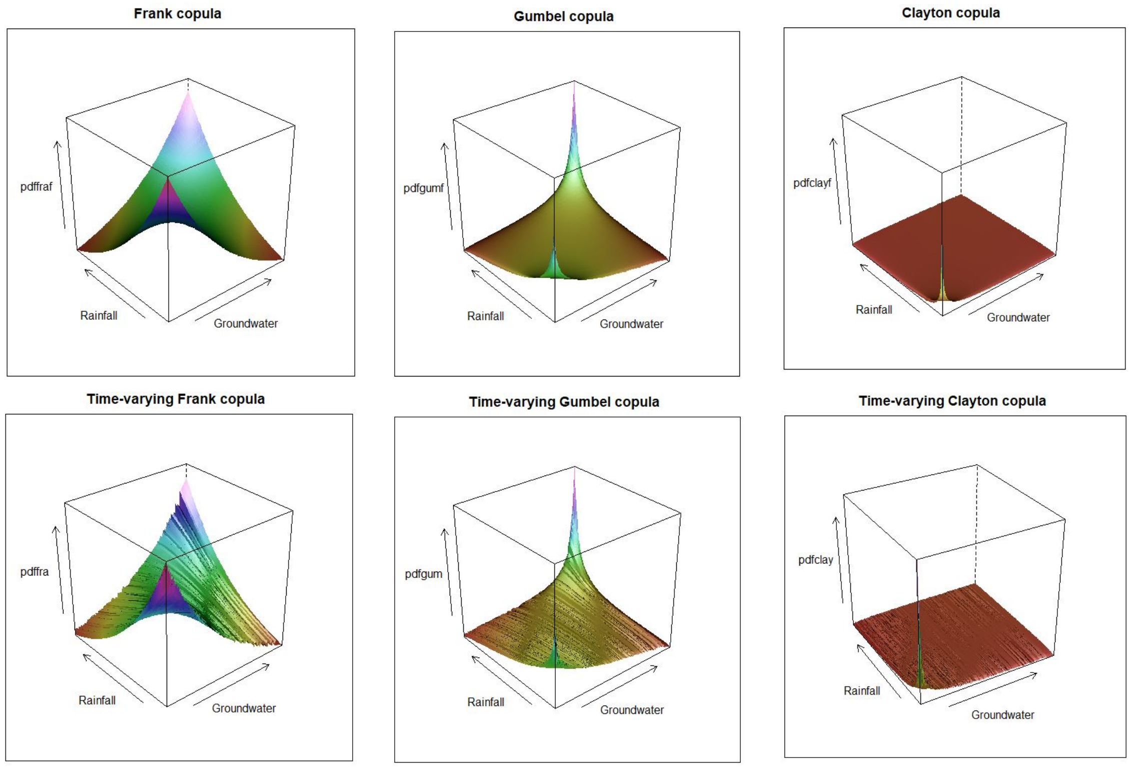

Figure 3. Wireframe plot of fitting fix parameters (top) and time-varying parameters (bottom) copulas to the rainfall and groundwater data

4. Conclusions

In this psper, we presented a method for fitting distributions in the case that the data are collected in a regular time structure and have autocorrelation. In such a case, the assumption of independence and identical distribution of the data is violated, and therefore it is incorrect to apply the maximum likelihood method to the data considering a fixed and identical parameter. In such a case, the time should also be considered and entered into the model through a regression function. Such models are called time-varying models. In this paper, polynomial regression and exponential function are used to enter the time factor into the model

References

[1] C. Almeida, C. Czado and H. Manner, Modeling high-dimensional time-varying dependence using dynamic D-vine models, *Appl. Stoch. Models Bus. Ind.*, **32** no. 5 (2016) 621–638.

[2] S. Amini, R. Z. Bidaki, R. Mirabbasi and M. Shafaei, Flood risk analysis based on nested copula structure in Armand Basin, Iran, *Acta Geophys.*, **70** (2022) 1385–1399.

https://dx.doi.org/10.22108/msci.2024.140292.1637

[3] Z. Azhdari, O. Bazrafshan, H. Zamani, M. Shekari and V. P. Singh, Hydro-meteorological drought risk assessment using linear and nonlinear multivariate methods, *Physics and Chemistry of the Earth*, Parts A/B/C, **123** (2021).

[4] B. L. Bowerman, R. T. O'Connell and A. B. Koehler *Forecasting, time series, and regression: an applied approach*, Thomson Brooks/Cole, Business & Economics, 2005, 686 pp.

[5] G. Casella and R. L. Berger, *Statistical inference*, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1990.

[6] H. Chowdhary, L. A. Escobar and V. P. Singh, Identification of suitable copulas for bivariate frequency analysis of flood peak and flood volume data, *Hydrology Research*, **1** (2011) 193–216.

[7] D. J. Dupuis, Using copulas in hydrology: Benefits, cautions, and issues, *J. Hydrol. Eng.*, **12** no. 4 (2007) 381–393.

[8] P. Jaworski, F. Durante, W. K. Hardle and T. Rychlik, *Copula theory and its applications*, Berlin, Springer, 2010.

[9] C. Jiang, L. Xiong, C. Y. Xu and S. Guo, Bivariate frequency analysis of nonstationary low-flow series based on the time-varying copula, *Hydrological Processes*, **15** no. 29 (2015) 1521–1534.

[10] C. M. Hafner and H. Manner, Dynamic stochastic copula models: Estimation, inference and applications, *J. Appl. Econometrics*, **27** no. 2 (2012) 269–295.

[11] S. Hesarkazzazi, R. Arabzadeh, M. Hajibabaei, W. Rauch, T. R. Kjeldsen, I. Prosdocimi, A. Castellarin and R. Sitzenfrei, Stationary vs non-stationary modelling of flood frequency distribution across northwest England, *Hydrological Sciences Journal*, **66** no. 4 (2021) 729–474.

[12] S. C. Kao and R. S. Govindaraju, A copula-based joint deficit index for droughts, *Journal of Hydrology*, **380** no. 1-2 (2010) 121–134.

[13] M. J. Machado, B. A. Botero, J. López, F. Francés, A. Díez-Herrero and G. Benito, Flood frequency analysis of historical flood data under stationary and non-stationary modelling, *Hydrol. Earth Syst. Sci.*, **19** no. 6 (2015) 2561–2576.

[14] H. Manner and O. Reznikova, A survey on time-varying copulas: specification, simulations, and application, *Econometric Rev.*, **31** no. 6 (2012) 654-687.

[15] A. A. Pathak and B. M. Dodamani, Connection between meteorological and groundwater drought with copula-based bivariate frequency analysis, *J. Hydrol. Eng.*, **26** no. 7 (2021) 05021015.

[16] A. J. Patton, Modelling asymmetric exchange rate dependence, *Internat. Econom. Rev.*, **47** no. 2 (2006) 527–556.

[17] M. Rodell, I. Velicogna and J. S. Famiglietti, Satellite-based estimates of groundwater depletion in India, *Nature*, **460** (2009) 999–1002.

[18] A. Sklar, Random variables, joint distribution functions, and copulas, *Kybernetika (Prague)*, **9** (1973) 449–460.

[19] D. M. Stasinopoulos and R. A. Rigby, Generalized additive models for location scale and shape (GAMLSS) in R, *Journal of Statistical Software*, **23** (2008) 1-46.

[20] M. Xing, D. Xu and J. He, Modeling method of failure dependent system based on time varying copula function, *Vibroengineering Procedia*, **21** (2017) 76-81.

Hossein Zamani

Department of Statistics, University of Hormozgan BandarAbbas, Iran

Email: zamani.huni hormozgan.ac.ir

Zohreh Pakdaman

Department of Statistics, University of Hormozgan BandarAbbas, Iran

Email: zpakdaman hormozgan.ac.ir

Marzieh Shekari

Department of Statistics, University of Hormozgan BandarAbbas, Iran

Email: shekarimuni hormozgan.ac.ir

https://dx.doi.org/10.22108/msci.2024.140292.1637