http://math-sci.ui.ac.ir

Journal of Mathematics and Society
ISSN (print): 2345-6493, ISSN (on-line): 2345-6507
Vol. 9 No. 3 (2024), pp. 93-106.

http://www.ui.ac.ir

THE CONCEPT AND PROPERTIES OF QUASI-RIGID C_0-SEMIGROUPS

MANSOOREH MOOSAPOOR

Abstract. In this paper, quasi-rigid C_0-semigroups are introduced and their properties are investigated. Also, some sufficient conditions for quasi-rigidity are stated. It is proved that quasi-rigid C_0-semigroups are recurrent. It is stated that if a C_0-semigroup contains a quasi-rigid operator, then it is quasi-rigid. The concept of topologically quasi-rigid C_0-semigroups is introduced. It is proved that topologically quasi-rigidity and quasi-rigidity are equivalent. The direct sum of quasi-rigid C_0-semigroups is investigated. It is demonstrated that a C_0-semigroup is quasi-rigid if and only if the direct sum of the C_0-semigroup with itself is quasi-rigid. Moreover, the finite direct sum of a quasi-rigid operator with itself is recurrent. So, quasi-rigidity with respect to the recurrency is similar to the weakly-mixing with respect to the hypercyclicity.

1. Introduction

Let X be a Banach space. An operator T on X is called topologically transitive if for every non-empty open sets U and V of X, there exists a natural number n such that $T^n(U) \cap V \neq \phi$ and it is called recurrent if for any non-empty open subset U of X, there exists a natural number n such that $T^n(U) \cap U \neq \phi$ [7]. If there exists $x \in X$ such that $orb(T, x) = \{x, Tx, T^2x, \ldots\}$ is dense in X, then T is called hypercyclic. On separable Banach spaces hypercyclicity and toplogical transitivity

Keywords: Quasi-rigid C_0-semigroups, Rigid C_0-semigroups, Recurrent C_0-semigroups, Direct sum

Article Type: Research Paper.

Communicated by Alireza Abdollahi.

Received: 03-10-2023, Accepted: 13-04-2024, Published Online: 24-11-2024.

Cite this article: M. Moosapoor, The concept and properties of quasi-rigid C_0-semigroups, *Journal of Mathematics and Society*,**9** no. 3 (2024) 93-106.

http://dx.doi.org/10.22108/msci.2024.139342.1611 .

are equivalent [10]. Also, an operator T on X is called rigid with respect to increasing sequence (n_k) of natural numbers if $T^{n_k}x \to x$ for any $x \in X$ [8].

The concept of quasi-rigid operators is recently introduced in [9]. An operator T on X is called quasi-rigid with respect to increasing sequence (n_k) of natural numbers if there exists a dense subset M of X such that $T^{n_k}x \to x$, for any $m \in M$.

A family $(T_t)_{t\geq 0}$ of operators on X is called a C_0-semigroup on X if $T_0 = I$ and for every $x \in X$ and $t, s \geq 0$,

$$T_{t+s} = T_t T_s \quad \text{and} \quad \lim_{s\to t} T_s x = T_t x.$$

A C_0-semigroup $(T_t)_{t\geq 0}$ on X is named topologically transitive if for every non-empty open sets U and V of X, there exists a real number $t > 0$ such that $T_t(U) \cap V \neq \phi$ and it is named hypercyclic if there exists $x \in X$ such that $orb((T_t)_{t\geq 0}, x) = \{x, Tx, T^2x, \dots\}$ is dense in X [4]. These two concepts are equivalent on separable Banach spaces [10].

The direct sum of two Banach spaces is defined by $X \oplus Y = \{x \oplus y : x \in X \text{ and } y \in Y\}$. A C_0-semigroup $(T_t)_{t\geq 0}$ on X is called weakly-mixing if $(T_t \oplus T_t)_{t\geq 0}$ is hypercyclic on $X \oplus X$ [10].

Also, a C_0-semigroup $(T_t)_{t\geq 0}$ on X is called recurrent if for every non-empty open subset U of X, there exists a real number $t > 0$ such that $T_t(U) \cap U \neq \phi$ [13]. Also, $(T_t)_{t\geq 0}$ is called rigid if there exists an increasing sequence (t_k) of positive real numbers such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} T_{t_k} x = x$ for every $x \in X$ [8].

In this paper, quasi-rigid C_0-semigroups are introduced and their properties are investigated. Also, the direct sum of these C_0-semigroups and their relations with the recurrent C_0-semigroups are investigated.

2. Main Results

Definition 2.1. *Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X and assume (t_k) is an increasing sequence of positive real numbers such that $\lim_{k\to\infty} t_k = \infty$. Then $(T_t)_{t\geq 0}$ is called quasi-rigid with respect to (t_k), if there exists a dense subset Y of X such that for every $x \in Y$, $T_{t_k}x \to x$.*

Two examples are presented in the following.

Example 2.2. *Let $a \in \mathbb{R}$. Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on $\mathbb{C}$ such that $T_t = e^{iat}I$ for any $t \geq 0$. Then by [8, Remark 2.7], $(T_t)_{t\geq 0}$ is rigid and so, it is quasi-rigid.*

Example 2.3. *Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on $\mathbb{C}$ such that $T_t = \lambda_t I$ for any $t \geq 0$ with this property that for any $t, s \geq 0$, $\lambda_s \lambda_t = \lambda_{t+s}$ and $|\lambda| = 1$. Hence, $(T_t)_{t\geq 0}$ is quasi-rigid; since for every $x \in \mathbb{C}$ and for every increasing sequence (t_k), of positive real numbers such that $\lim_{k\to\infty} t_k = \infty$,*

$$T t_k x = \lambda_{t_k} x \to x.$$

http://dx.doi.org/10.22108/msci.2024.139342.1611

The following lemma shows that the set of quasi-rigid C_0-semigroups is a subset of the set of recurrent C_0-semigroups.

Lemma 2.4. *If $(T_t)_{t\geq 0}$ is a quasi-rigid C_0-semigroup, then it is recurrent.*

Theorem 2.5. *For a C_0-semigroup $(T_t)_{t\geq 0}$ on a Banach space X, if $(T_s$ is quasi-rigid for some $s>0$, then $(T_t)_{t\geq 0}$ is quasi-rigid.*

Corollary 2.6. *Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X. If there exists $s>0$ such that T_s is power bounded and*

$$X=\overline{lin}\{x\in X|Tx=\lambda x;\ \textit{for a scalar}\ \lambda\in\mathbb{C}\ \textit{with}\ |\lambda|=1\},$$

then $(T_t)_{t\geq 0}$ is quasi-rigid.

Two C_0-semigroup $(T_t)_{t\geq 0}$ on a Banach space X and $(S_t)_{t\geq 0}$ on a Banach space Y are called quasiconjugate if there exists an operatr $\phi : X\to Y$ with dense range such that $S_t\circ\phi=\phi\circ T_t$ for any $t\geq 0$ [10, Definition 7.11].

Theorem 2.7. *Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X and assume $(S_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space Y. If $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are quasiconjugate, then quasi-rigidity of $(T_t)_{t\geq 0}$ implies that $(S_t)_{t\geq 0}$ is quasi-rigid.*

Definition 2.8. *Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X and assume (t_k) is an increasing sequence of positive real numbers such that $\lim_{k\to\infty}t_k=\infty$. Then $(T_t)_{t\geq 0}$ is called topologically quasi-rigid with respect to (t_k) if for every non-empty open subset U of X, there exists $k_U>0$ such that $T_{t_k}(U)\cap U\neq\phi$ for any $k\geq k_U$.*

Theorem 2.9. *Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X and assume (t_k) is an increasing sequence of positive real numbers. Then the following are equivalent:*

(i) *$(T_t)_{t\geq 0}$ is quasi-rigid with respect to (t_k).*
(ii) *$(T_t)_{t\geq 0}$ is topologically quasi-rigid with respect to (t_k).*

Theorem 2.10. *Assume $(S_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X and assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space Y. Then if $(S_t\oplus T_t)_{t\geq 0}$ is quasi-rigid on $X\oplus Y$, then $(S_t)_{t\geq 0}$ is quasi-rigid on X and $(T_t)_{t\geq 0}$ is quasi-rigid on Y.*

Corollary 2.11. *Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X. If $(T_t\oplus T_t)_{t\geq 0}$ is a quasi-rigid C_0-semigroup, then $(T_t)_{t\geq 0}$ is quasi-rigid. Especially, $(T_t)_{t\geq 0}$ is recurrent.*

Theorem 2.12. *If $(T_t)_{t\geq 0}$ is a quasi-rigid C_0-semigroup on X, then $(T_t\oplus T_t)_{t\geq 0}$ is quasi-rigid. Especially, $(T_t\oplus T_t)_{t\geq 0}$ is recurrent.*

http://dx.doi.org/10.22108/msci.2024.139342.1611

By Corollary 2.11 and Theorem 2.12, the consequence result is stated.

Corollary 2.13. *Suppose $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X. Then $(T_t)_{t\geq 0}$ is quasi-rigid if and only if $(T_t \oplus T_t)_{t\geq 0}$ is quasi-rigid.*

Theorem 2.14. *Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X and assume $(S_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space Y. Suppose (t_k) is an increasing sequence of positive real numbers such that $\lim_{k\to\infty} t_k = \infty$. If $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are quasi-rigid with respect to (t_k), then $(T_t \oplus S_t)_{t\geq 0}$ is quasi-rigid with respect to (t_k). Especially, $(T_t)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ are recurrent.*

Theorem 2.15. *Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X. Then the following are equivalent:*

(i) *$(T_t)_{t\geq 0}$ is quasi-rigid with respect to (t_k).*
(ii) *$(T_t \oplus T_t)_{t\geq 0}$ is recurrent.*
(iii) *$(T_t \oplus T_t \oplus \cdots \oplus T_t)_{t\geq 0}$ is recurrent.*

Using Theorem 2.9 and Theorem 2.15, lead to the following corollary.

Corollary 2.16. *Assume $(T_t)_{t\geq 0}$ is a C_0-semigroup on a Banach space X. Then the following are equivalent:*

(i) *$(T_t)_{t\geq 0}$ is quasi-rigid with respect to (t_k).*
(ii) *$(T_t)_{t\geq 0}$ is topologically quasi-rigid with respect to (t_k).*
(iii) *$(T_t \oplus T_t \oplus \cdots \oplus T_t)_{t\geq 0}$ is recurrent.*

3. Conclusions

The concept of the quasi-rigidity for C_0-semigroups is a generalization of the concept of rigidity for C_0-semigroups. As it is proved in this paper, this concept is equivalent to the concept of topologically quasi-rigid.

Quasi-rigid C_0-semigroups are recurrent. Also, the finite direct sum of a quasi-rigid C_0-semigroup with itself is recurrent. Quasi-rigidity with respect to recurrency is similar to weakly-mixing with respect to hypercyclicity. Moreover, it is proved that if the direct sum of two C_0-semigroups is quasi-rigid, then any of them is quasi-rigid and under some conditions the converse is true. Consequently, a C_0-semigroup is quasi-rigid if and only if the direct sum of it to itself is quasi-rigid.

References

[1] M. Amouch and O. Benchiheb, On a class of super-recurrent operators, *Filomat*, **36** no. 11 (2022) 3701–3708.

http://dx.doi.org/10.22108/msci.2024.139342.1611

[2] N. Bamerni, A. Kilicman and M. S. Md. Noorani, A review of some works in the theory of diskcyclic operators, *Bull. Malays. Math. Sci. Soc.*, **39** no. 2 (2016) 723–739.

[3] F. Bayart and E. Matheron, *Dynamics of linear operators*, Cambridge Tracts in Mathematics, **179**, Cambridge University Press, Cambridge, 2009.

[4] L. Bernal-Gonzalez and K.G. Grosse-Erdmann, Existence and nonexistence of hypercyclic semigroups, *Proc. Amer. Math. Soc.*, **135** no. 3 (2007) 755–766.

[5] J. Bes, Q. Menet, A. Peris and Y. Puig, Strong transitivity properties for operators, *J. Differential Equations*, **266** no. 2-3 (2019) 1313–1337.

[6] J. A. Conejero, V. Müller and A. Peris, Hypercyclic behaviour of operators in a hypercyclic C_0-semigroup, *J. Funct. Anal.*, **244** no. 1 (2007) 342–348.

[7] G. Costakis, A. Manoussos and I. Parissis, Recurrent linear operators, Recurrent linear operators, *Complex Anal. Oper. Theory*, **8** no. 8 (2014) 1601–1643.

[8] T. Eisner and S. Grivaux, Hilbertian Jamison sequences and rigid dynamical systems, *J. Funct. Anal.*, **261** no. 7 (2011) 2013–2052.

[9] S. Grivaux, A. Lopez-Martinez and A. Peris, Questions in linear recurrence: From the $T \oplus T$-problem to lineability, (2022), arXiv:2212.03652.

[10] K. G. Grosse-Erdmann and A. Peris Manguillot, *Linear Chaos*, Springer-Verlag London Ltd., London, 2011.

[11] M. Moosapoor, Diskcyclic C_0-Semigroups and diskcyclicity criteria, *Nonlinear Functional Analysis and Applications*, **27** no. 1 (2022) 111–119. https://doi.org/10.22771/nfaa.2022.27.01.07

[12] M. Moosapoor, Supermixing and hypermixing of strongly continuous semigroups and their direct sum, *J. Taibah Univ. Sci.*, **15** no. 1 (2021) 953–959.

[13] M. Moosapoor, On the Recurrent C_0-Semigroups, Their Existence, and Some Criteria, *J. Math.*, **2021** (2021) 1–7.

[14] T. K. S. Moothathu, Linear Independence of a Hypercyclic Orbit for Semigroups, *J. Math. Anal. Appl.*, **467** (2018) 704–710.

Mansooreh Moosapoor

Associate Professor, Department of Mathematics Education, Farhangian University, P.O.Box 14665-889, Tehran, Iran

Email: m.mosapourcfu.ac.ir