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RELATIONS BETWEEN THE DISTINGUISHING NUMBER AND SOME OTHER GRAPH PARAMETERS

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Abstract. A distinguishing coloring of a simple graph G is a vertex coloring of G which is preserved only by the identity automorphism of G. In other words, this coloring "breaks" all symmetries of G. The distinguishing number $D(G)$ of a graph G is defined to be the smallest number of colors in a distinguishing coloring of G. This concept of "symmetry breaking" was first proposed by Babai in 1977 and after the publication of a seminal paper by Albertson in 1996, it attracted the attention of many mathematicians. In this paper, along with studying some relations between $D(G)$ and some other important graph parameters, we introduce the concept of a (D, α)-ordinary graph which displays the comparison between $D(G)$ and the independence number $\alpha(G)$. Then we consider the classification of (D, α)-ordinary graphs in various families of graphs such as forests, cycles, generalized Johnson graphs, Cartesian products of graphs and line graphs of connected claw-free graphs. We also propose some conjectures and discuss about some future research directions in this topic.

1. Introduction

The graphs in the paper are assumed to be undirected, simple and finite. For a graph G, we denote by $V(G)$ and $\mathrm{Aut}(G)$, the vertex set and the automorphism group of G, respectively. Let G be a graph and c be a vertex (respectively, edge) coloring of G. We say that an $\alpha \in \mathrm{Aut}(G)$ preserves the

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coloring c if for any pair of vertices u and v in $V(G)$ (respectively, any pair of edges e and f), for which $\alpha(u) = v$ (respectively, $\alpha(e) = f$), the vertices u and v (respectively, the edges e and f) have the same color. For a graph G, a *distinguishing coloring* (respectively, *distinguishing edge coloring*) is a vertex coloring (respectively, edge coloring) of G such that the only automorphism which preserves the coloring, is the identity automorphism. Then the *distinguishing number* (respectively, *distinguishing index*) of G, denoted by $D(G)$ (respectively, $D'(G)$), is defined to be the smallest number of colors for a distinguishing coloring (respectively, distinguishing edge coloring) of G. It is easy to see that $D(K_n) = n$, $D(K_{n,n}) = n+1$, $D(P_n) = 2$ for $n \geq 2$, $D(C_3) = D(C_4) = D(C_5) = 3$, while $D(C_n) = 2$, for $n \geq 6$. Note also that, for any asymmetric graph G, we have $D(G) = D'(G) = 1$ and, as mentioned in [2], $D'(K_{n,n}) = 2$, for $n \geq 4$. This symmetry breaking concept was first introduced by Babai in 1977, and after 1996, it attracted the attention of many mathematicians. For detailed studies on this topic, we refer the reader to [1], [3]-[15].

In [1], along with some new parameters related to distinguishing colorings, the authors introduced the *distinguishing threshold* $\theta(G)$ as the minimum number k of colors such that any coloring of the graph G with k colors is distinguishing. Obviously $D(G) \leq \theta(G) \leq |V(G)|$. They also showed that

$$\theta(K_n) = \theta(\overline{K_n}) = n,\ \theta(K_{m,n}) = m+n,\ \theta(P_n) = \lceil \frac{n}{2} \rceil + 1,\ \text{for } n \geq 2,$$

and $\theta(C_n) = \lfloor \frac{n}{2} \rfloor + 2$, for $n \geq 3$.

It has been shown that the distinguishing number of "almost" all connected graphs is at most 2. On the other hand, the independence number of a graph G is at least 2 unless G is a complete graph on more than 1 vertices. It follows that for almost all connected graphs G, $D(G) \leq \alpha(G)$. This is a motivation for us to define the concept of (D,α)-*ordinary* graphs: a graph is called (D,α)-ordinary if it satisfies the above inequality. In this paper, we consider this concept and determine families of graphs which are ordinary or non-ordinary. In particular we analyse the complete multipartite graphs, Cartesian product of graphs and some line graphs.

2. Main Results

In this section, we list the main results of the paper without proofs. We start by the following two general results in which we use the notation $\mathrm{Fix}(G)$ for the *fixing number* of a graph G, i.e. the smallest size of a *fixing set* of G, where a fixing set of G is a subset of vertices of G which is point-wise preserved only by the identity automorphism.

Proposition 2.1. *For each graph G, we have $D(G) \leq \mathrm{Fix}(G) + 1 \leq \theta(G)$.*

For the next result, we also recall the notion of $\beta(G)$, the *vertex cover number* of a graph G and recall that $\alpha(G) + \beta(G) = |V(G)|$.

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Theorem 2.2. *For any graph G, if $\theta(G) \neq |V(G)|$ then* $\mathrm{Fix}(G) < \beta(G)$.

Also, for the next result, we use the notion of Cartesian products and powers of graphs.

Theorem 2.3. *Let G be a connected graph. If $G \ncong K_2^2, K_2^3, K_3^2$, then for each $r \geq 2$, $D(G^r) = 2$, otherwise $D(G) = 3$.*

For example, for any $n \geq 3$ and $r \geq 3$, we have that $D(K_n^r) = 2$.

Throughout the paper, we also use the notation $L(G)$ of the line graph of a graph G and denote the graph obtained from the graph K_4 by removing two non-adjacent edges (respectively, one edge) by Q (respectively, R). See Figure 1. Note that $L(Q) = R$.

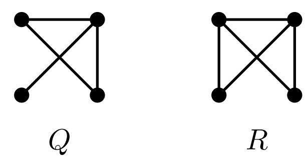

FIGURE 1. the graph Q and the graph R

In this part of the paper, we start investigating the ordinariness property of graphs. It is straightforward to see that, for $n \geq 6$, the graphs $\overline{K_n}$ and C_n, and for $n \geq 3$, the graph P_n, and for $m \neq n$, the graph $K_{m,n}$ are ordinary. The same is true for all asymmetric graphs, while the graphs such as K_n and C_4 are non-ordinary. We also observe that the set of all ordinary graphs are closed under the union operation while the set of non-ordinary graphs are closed under the "join" operation. Therefore there are infinite families of ordinary and non-ordinary graphs. This is an insight that the problem of determining such graphs is an interesting one.

Let $|V(G)| = n$. We note that, according to Proposition 2.2, for any graph G with $\theta(G) \neq n$, if $\alpha(G)$ is large, then $\mathrm{Fix}(G)$ must be small. Another consequence of Proposition 2.2 is the following which provides a necessary condition for a graph to be non-ordinary.

Proposition 2.4. *If G is non-ordinary, then $\theta(G) = n$ or $\alpha(G) \leq \frac{n-1}{2}$.*

Corollary 2.5. *For a graph G, if it holds that $\theta(G) \neq n$ and $\alpha(G) > \frac{n-1}{2}$, then $D(G) \leq \alpha(G)$.*

In what follows, we study some families of graphs from view-point of ordinariness. We start with forests, cycles and bipartite graphs.

Proposition 2.6. *Every forest with more that two vertices is ordinary.*

Proposition 2.7. *The cycle C_n is ordinary if and only if $n \geq 6$.*

Proposition 2.8. *All bipartite graphs are ordinary except the graphs $K_{n,n}$.*

We Also present necessary and sufficient conditions of ordinariness for the more general family of complete multipartite graphs.

Theorem 2.9. *Assume $G = K_{n_1^{j_1}, n_2^{j_2}, \ldots, n_r^{j_r}}$, in which $n_1 > n_2 > \cdots > n_r$ and, for any $i = 1, 2, \ldots, r$, there are j_i parts of size n_i in G. Then G is non-ordinary if and only if $j_1 > 1$, or there is an $i > 1$ in such a way that $\binom{n_1}{n_i} < j_i$.*

Furthermore, we can classify the ordinary generalized Johnson graphs as follows.

Theorem 2.10. *The graph $J(n, k, i)$ is ordinary if and only if (n, k, i) is neither $(4, 2, 1)$ nor $(5, 2, 1)$.*

We next consider the Cartesian products of graphs and start with the following observation.

Proposition 2.11. *Assume G and H are two relatively prime ordinary connected graphs which are not asymmetric. Then $G \square H$ is ordinary.*

In addition, we show that, except for some small cases, the Cartesian powers of connected graphs, with exponents at least 2, are ordinary.

Corollary 2.12. *For any graph G and any number $r \geq 2$, the graph G^r is ordinary unless $G^r \cong K_2^2$.*

Corollary 2.13. *Assume $G = G_1^{\alpha_1} \square \cdots \square G_r^{\alpha_r}$ to be the prime factorization of the graph G, where*

(1) *all G_is are connected;*
(2) *if $\alpha_i = 1$, then G_i is ordinary but not asymmetric, and*
(3) *$G \ncong K_2^2$.*

Then G is ordinary.

We now consider the main question of the paper for the line graphs.

Proposition 2.14. *The graph $L(K_n)$ is ordinary if and only if $n \geq 6$ or $n = 2$.*

For what follows, we define the family $\mathcal{A} = \{P_3, K_3, R, K_4, C_4, C_5, K_5\}$. One easily observes the following.

Lemma 2.15. *The graphs in $\mathcal{A}$ are non-ordinary.*

We now classify the line graphs of claw-free graphs with respect to the property of ordinariness. More precisely, we show that the members of $\mathcal{A}$ are the only connected claw-free graphs whose line graphs are non-ordinary.

Theorem 2.16. *Assume G to be a connected claw-free graph. The graph $L(G)$ is non-ordinary if and only if $G \in \mathcal{A}$.*

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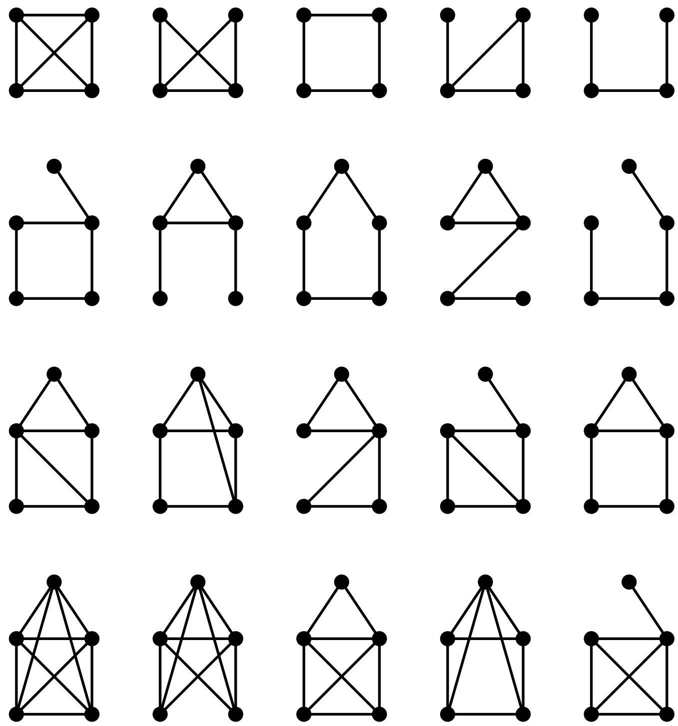

FIGURE 2. The twenty graphs discussed in the proof of theorem 2.16

In the next result, we make use of the concept of *iterated* line graphs. More clearly, we use the notation $L^1(G) = L(G)$ and $L^r(G) = L(L^{r-1}(G))$, for $r \geq 2$.

Corollary 2.17. *Let G be a connected graph for which $L(G) \notin \{K_3, Q, R, C_4, K_4, C_5, K_5\}$. Then for any $r \geq 2$, the graph $L^r(G)$ is ordinary unless $L(G) = P_n$, in which case the corollary holds for $n \leq 3 + r$.* $\square$

We finally, consider the line graphs of complete multipartite graphs.

Theorem 2.18. *Let $G = K_{n_1^{j_1}, n_2^{j_2}, \ldots, n_r^{j_r}}$, where $n_1 > n_2 > \cdots > n_r$ and, for each $i = 1, 2, \ldots, r$, there are j_i parts of size n_i in G. Assume $\Sigma = \sum_{i=2}^{r} j_i n_i + (j_1 - 1)n_1$. Then $L(G)$ is ordinary if and only if $n_1 \leq \Sigma^{\Sigma}$, unless $G \cong R$, in which case $L(G)$ is non-ordinary.*

3. Conclusions

We investigated some families of graphs; including complete multipartite graphs, Cartesian products and some line graphs; to see under which conditions they are ordinary. These are only some small portion of graphs and it seems an interesting problem to consider other families of graphs and find necessary and/or sufficient conditions for their ordinariness.

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