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GENERALIZED DERIVATIONS ON CERTAIN BANACH ALGEBRAS

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Abstract. In this paper, we apply the well-known results concerning derivations and generalized derivations of commutative Banach algebras and of prime rings to certain Banach algebras that are neither commutative Banach algebras nor prime rings. For example, we investigate the truth of Singer-Wermer conjecture and Posner's second theorem for this class of Banach algebras.

1. Introduction

A well-known theorem of Singer and Wermer states that every bounded derivation on a commutative Banach algebra has its image in the radical [14]. About 30 years later, Thomas extended the Singer-Wermer theorem to arbitrary, not necessarily bounded, derivations [15]. A number of authors have generalized this theorem in several ways on certain Banach algebras; see [11, 13]. For example, Posner [13] gave a noncommutative version of the Singer-Wermer theorem for prime rings. Also he proved that the zero map is the only centralizing derivation on a noncommutative prime ring (Posner's second theorem).

In harmonic analysis, some examples of Banach algebras have been discovered for which $\operatorname{rad}(\mathcal{A}) = \operatorname{rann}(\mathcal{A})$ and the algebra $\mathcal{A}/\operatorname{rad}(\mathcal{A})$ is commutative. The most notable cases are $L_0^\infty(G)^*$ when G is

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an abelian locally compact group and $L_0^\infty(G)$ is the subspace of $L^\infty(G)$ consisting of all functions $f \in L^\infty(G)$ vanishing at infinity [10] and $VN(G)^*$ when G is a discrete group and $VN(G)$ is the von Neumann algebra generated by the left regular representation of G [9]. Moreover, under the right conditions, all of the above examples are neither commutative Banach algebras nor prime rings. These motivated us to try to apply the well-known results concerning derivations of commutative Banach algebras and derivations of prime rings to this class of Banach algebras. For example, we prove that the Singer-wermer theorem and Posner's second theorem for derivations and general derivations hold on this class of Banach algebras. Our results generalize and unify most of the results of [1, 12] concerning the convolution algebra $L_0^\infty(G)^*$ over an abelian locally compact group G.

2. Main Results

Throughout this paper, $\mathcal{A}$ denotes a Banach algebra with the properties that $\mathrm{rad}(\mathcal{A}) = \mathrm{rann}(\mathcal{A})$ and the algebra $\mathcal{A}/\mathrm{rad}(\mathcal{A})$ is commutative.

Theorem 2.1. *Let d be a derivation of $\mathcal{A}$. Then $d(\mathcal{A}) \subseteq \mathrm{rad}(\mathcal{A})$.*

Before giving the following consequence of Theorem 2.1, let us recall that a linear mapping $T : \mathcal{A} \to \mathcal{A}$ is called spectrally bounded if there exists $M \geq 0$ such that $r(T(a)) \leq Mr(a)$ for all $a \in \mathcal{A}$, where $r(a)$ denotes the spectral radius of $a \in \mathcal{A}$. In addition, if $M = 0$, T is called spectrally infinitesimal.

Corollary 2.2. *The following statements hold.*

(i) *The composition of two derivations of $\mathcal{A}$ is always a derivation of $\mathcal{A}$.*
(ii) *Every derivation of $\mathcal{A}$ is spectrally infinitesimal.*

Theorem 2.3. *If $\mathcal{A}$ has a right identity, then every generalized derivation of $\mathcal{A}$ is spectrally bounded.*

Theorem 2.4. *Let (δ, d) be a generalized derivation of $\mathcal{A}$. If $\mathcal{A}$ has a right identity, then the following statements are equivalent.*

(i) $\delta(\mathcal{A}) \subseteq rad(\mathcal{A})$.
(ii) *δ is spectrally infinitesimal.*
(iii) $\delta = d$.

We recall that the algebraic center of $\mathcal{A}$ is denoted by $Z(\mathcal{A})$. Let k be a fixed positive integer. A mapping $T : \mathcal{A} \to \mathcal{A}$ is called k-centralizing (resp. k-commuting) if $[T(a), a^k] \in Z(\mathcal{A})$ (resp. $[T(a), a^k] = 0$) for all $a \in \mathcal{A}$. In the case $k = 1$, T is said to be centralizing (resp. commuting).

Theorem 2.5. *Let d be a derivation of $\mathcal{A}$ and let k be a positive integer. If $\mathcal{A}$ has a right identity, then the following statements are equivalent.*

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(i) $d = 0$.

(ii) d *is* k*-centralizing.*

(iii) d *is* k*-commuting.*

Theorem 2.6. *Let* (δ, d) *be a generalized derivation of* $\mathcal{A}$ *and* $k \in \mathbb{N}$*. If* $\mathcal{A}$ *has a right identity, then the following statements are equivalent.*

(i) δ *is* k*-commuting.*

(ii) δ *is* k*-centralizing.*

(iii) δ *is a right multiplier.*

(iv) *There exists* $b \in \mathcal{A}$ *such that* $\delta = R_b$.

The maps T and S from $\mathcal{A}$ into $\mathcal{A}$ are called orthogonal, denoted by $T \perp S$, if $T(a)cS(b) = S(b)cT(a) = 0$ for all $a, b, c \in \mathcal{A}$.

Theorem 2.7. *Let* (δ, d) *be a generalized derivation of* $\mathcal{A}$*. If* $\mathcal{A}$ *has a right identity, then the following statements are equivalent.*

(i) $[[\delta(a), a], \delta(a)] \in Z(\mathcal{A})$ *for all* $a \in \mathcal{A}$.

(ii) $d \perp \delta$.

(iii) (δ^2, d^2) *is a generalized derivation of* $\mathcal{A}$.

3. Conclusions

Let $\mathcal{A}$ be a Banach algebra with the properties that $\mathrm{rad}(\mathcal{A}) = \mathrm{rann}(\mathcal{A})$ and the algebra $\mathcal{A}/\mathrm{rad}(\mathcal{A})$ is commutative. We show that a derivation of $\mathcal{A}$ maps $\mathcal{A}$ into $\mathrm{rad}(\mathcal{A})$. Using this, we determine among other things when a generalized derivation of $\mathcal{A}$ maps $\mathcal{A}$ into $\mathrm{rad}(\mathcal{A})$. We also study k-centralizing generalized derivations of $\mathcal{A}$. Then, for a generalized derivation (δ, d) of $\mathcal{A}$ we obtain a necessary and sufficient condition for (δ^2, d^2) to be still a generalized derivation. The main applications are concerned with the algebras over locally compact groups. In particular, we deduce these results for bidual of Fourier algebras of discrete amenable groups as an application of our approach.

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