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SOLITON AND ITS APPLICATION IN THE STUDY OF NONLINEAR DYNAMICS OF ACOUSTIC WAVES IN MULTI-PARTICLE FLUID

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Abstract. Solitons have many applications in pure and applied mathematics, especially in the fields such as nonlinear differential equations, Lie algebra, and algebraic geometry. Solitons are ubiquitous in nature and have many applications in nonlinear dynamics. The discovery of solitons made it possible to obtain analytical solutions for nonlinear differential equations. Among them Korteweg-de Vries equation is the most familiar one, which is the most canonical nonlinear wave equation. In an electrostatic fluid that is partially ionized, the collision of ions with each other and also with electrons cause the propagation and coupling of ionic electrostatic waves to behave nonlinear. The equations describing the dynamics of acoustic waves in this fluid are transformed into the Korteweg-de Vries equation by the reduction perturbation method, which analytical solutions are in the form of solitary waves, and their amplitude and speed depend on the properties of the fluid medium. A brief history of solitons and their applications are given. Then, the nonlinear propagation of the traveling ion electrostatic wave in a multi-particle fluid is obtained by taking into account the production-loss rate of ions. The analytical and numerical solution of the wave equations is obtained. The results show that the wave dispersion curves have bifurcation points and only high-frequency waves are propagated as solitary waves with complex modes. The results of this study can be used in the observation of ion acoustic waves in space plasmas and the presence of dust grains in laboratory plasma.

Keywords: non linear dynamics, bifurcation,complex dispersion equation, acoustic wave.

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1. Introduction

The history of solitons or soliton waves backs to the early 19th century. It was first discovered by John Scott Russel, a young Scottish engineer in the Victorian shipbuilding industry, and he called them the great wave of translation [1]. He conducted experiments to find the most efficient channel design for ships and boats. Russell's observations in Hermiston Union Canal, near Riccarton campus, Heriot-Watt University, Edinburgh, led to a very important scientific discovery. He presented his achievement under the title "Report on Waves" at the 14th meeting of the British Association for the Advancement of Science in September 1844 [2]. Theoretical definitions of solitary waves were presented by Boussinesq in 1871 and Riley in 1876 [1]. Nevertheless, the debate about solitons continued until 1895 when Korteweg and de Vries presented their famous equation called KdV [3].

The importance of soliton waves was finally shown in 1965 by Zabosky and Kruskal, who numerically investigated the KdV equation and observed quasi-particle behavior in it. The special aspect that they observed was that the solitary waves retain their shape and speed after the collision, and they gave the name soliton to these waves [4, 5]. A series of integrable dynamical systems have soliton solutions obtained by the inverse scattering transform (IST), and it can be considered as the nonlinear equivalent of the Fourier transform method for linear differential equations.

One of the properties of the integrable equations with solitonary wave solution is that they have an infinite number of stability laws. Another feature is the existence of Hamiltonian structures that allow the system to be described without any need to obtain the exact solutions of the equations. Solitons in Hamilton's theory led to the emergence of new concepts and theories in many fields of mathematicas and physical sciences [1]. In the field of fluid dynamics, plasma, nonlinear optics, astrophysics and molecular biology, soliton theory has many applications. In optical fibers, soliotons play the role of transmitting digital signals over long distances. In biology, soliton theory is used to describe energy and signal propagation in biomembranes, the nervous system, and low-frequency motions in proteins and DNA [6]. The role of solitons is also seen in plasma physics studies, a substance that contains various charged particles [7].

In this article, the instability conditions and nonlinear behavior of ion acoustic waves in a fluid with partially ionized particles (different ion species and Boltzmann electrons) are investigated. Using the fluid equations, and converting it to the Korteweg-de Vries equation, soliton responses have been determined to describe the growth and propagation conditions of acoustic-ion modes.

2. Main Results

We consider an ion-electron multiparticle fluid medium where ions have low energy and electrons have a Boltzmann distribution. The basic equations to describe the dynamics of this ionic fluid in

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one-dimensional coordinates are as follows

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x}(\rho_i\, u_i) = 0 \tag{2.1a}$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} = -\frac{q_i}{m_i}\frac{\partial \varphi}{\partial x} - \nu_i \frac{u_i}{\rho_i} \tag{2.1b}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{e\rho_e}{\epsilon_0 m_e} - \frac{q_i \rho_i}{\epsilon_0 m_i} \tag{2.1c}$$

where ρ_i, u_i, m_i, q_i, and ν_i are the density, fluid velocity, mass and electric charge of ion i, and the rate of ion production and loss due to ionization and particle collisions per unit volume, respectively. The presence of electrons in this environment is given by the Poisson's equation, where φ is the electrostatic potential.

By expanding the dependent variables in the equations (2.1a) to (2.1c) as follows

$$\begin{aligned}
\rho_i &= \rho_0 + \varepsilon\rho_{i1} + \varepsilon^2\rho_{i2} + \varepsilon^3\rho_{i3} + \cdots\,, \\
u_i &= u_{i0} + \varepsilon u_{i1} + \varepsilon^2 u_{i2} + \varepsilon^3 u_{i3} + \cdots\,, \\
\phi &= \varepsilon\phi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\phi_3 + \cdots\,, \\
\nu_i &= \varepsilon\nu_{i1} + \varepsilon^2\nu_{i2}.
\end{aligned} \tag{2.2}$$

and substituting the wave solution $\exp(jkx - j\omega t)$ in the equations, we can obtain the dispersion relation of the excited wave. In the linear regime, the standard method of linearization gives the dispersion equation of the form

$$k^2\Big(\frac{\mu_i^2\rho_0^3}{(\omega - k\, u_{i0})^2 + \nu_{i1}^2} - 1\Big) = 1 \tag{2.3}$$

For $u_0 \geq 2.0$ and long wavelengths, the slow mode (low frequency mode) bifurcates. This shows how the nature of the answers of the equations depends on the parameters of that equation [8]. Unlike high frequency modes, low frequency modes remain almost unchanged at $\delta = \nu_1/\nu_2 \geq 1.0$ and short wavelengths, $k \geq 2.5$. In this limit, although the high frequency modes increase with the wave number k, the low frequency modes tend to zero. High and low frequency modes are fast and slow modes respectively. Also, in the limit of short wavelengths, the value of δ does not affect the frequency of both modes. Also, unlike the fast mode, the speed u_0 has the effect of reducing the frequency of the slow modes.

Let us now to proceed with the nonlinear regime of the hydrodynamical equations (2.1a) to (2.1c). By introducing two new variables

$$\xi = \varepsilon^{1/2}(x - \lambda t), \qquad \eta = \varepsilon^{3/2} x \tag{2.4}$$

and using the equation (2.2), after some algebra, the following variable-coefficient Kortweg-de Vries (vKdV) equation is obtained

$$\frac{\partial \phi_1}{\partial \eta} + [\frac{\mu_i^2 \rho_0}{2\lambda(u_{i0}-\lambda)^4}]\phi_1 \frac{\partial \phi_1}{\partial \xi} + \frac{1}{2\lambda^2}\frac{\partial^3 \phi_1}{\partial \xi^3} + [\frac{\mu_i^3 \nu_i}{2\lambda^3(u_{i0}-\lambda)^4}]\phi_1 = 0, \tag{2.5}$$

In general, this equation is not integrable and must be solved numerically. However, it is possible to obtain approximate analytical solutions for it [9]. To find the traveling solitary wave solutions, it is necessary that the wave profile is stationary with the form of $\phi(\zeta)$ with $\zeta = (\xi - \kappa\eta)$ and constant κ [10]. Therefore, equation (2.5) can be cast into the form

$$\frac{d^3\phi_1}{d\zeta^3} + \alpha\frac{d\phi_1}{d\zeta} + \beta\phi_1\frac{d\phi_1}{d\zeta} = 0, \tag{2.6}$$

where

$$\begin{aligned} \alpha &= 2\kappa\,\lambda^2 \\ \beta &= \frac{\mu_i^2 \lambda \rho_0}{(u_{i0}-\lambda)^4}\,\exp\Big(-\frac{\mu_i^3 \nu_i}{2\lambda^3(u_{i0}-\lambda)^4}\,\eta\Big) \end{aligned} \tag{2.7}$$

An analytical solution of the wave equation takes the form,

$$\phi(\zeta) = \phi_1\, sech^2\Big(\frac{1}{2}\zeta\sqrt{\frac{\phi_1}{3\beta}}\Big), \tag{2.8}$$

Tis is the hyperbolic soliton. The results show that the amplitudes of solitons decreases with increasing values of λ and ν. With the propagation of soliton waves to the areasof higher density, the amplitude of the waves increases. In the case of both negative and positive ions, for $\lambda = 0.5$ there is a combination of solitons with positive and negative polarization in the response of the equation. For $\lambda \geq 1.0$ there are only solitons with negative polarization.

3. Conclusions

Nonlinear differential equations are the mathematical description of many phenomena and physical processes in nature. In this research, the application of solitons as a solution to nonlinear differential equations governing the physics of fluids was described. The nonlinear propagation of sound waves in a multiparticle fluid (consisting of electrons and several types of ion) was studied and investigated using hydrodynamic equations, accounting for the rate of production and loss of ions. The study on the stability condiotion showed that bifurcation in the frequency of wave modes occur for nearly all values of fluid speed, u_0, and production-loss contribution, δ.

With the reductive perturbation method, the nonlinear hydrodynamic equations with a static profile is converted into the Korteweg-de Vries equation with variable coefficients. The analytical solution of this wave equation shows that the disturbance in the medium of a multi-particle fluid is propagated as a soliton wave. Numerical studies show that for the fluid o two positive ions, the polarization of the solitary wave is positive, the amplitude of the solitons decreases with the wave speed while

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its width increases. When a fluid contains both negative and positive ions, the polarization of the solitons becomes negative. Also, if the non-linear term of the wave equation becomes negligible, the combination of solitons with different polarizations is created. As the waves propagate towards areas with higher density, the amplitude of solitary waves increases and its speed decreases.

The obtained theoretical results can be used to observe electrostatic waves in space plasmas, as well as excitation of soliton waves, especially ion acoustic waves in laboratory plasmas. In addition, changing the phase velocity of ion acoustic waves is a diagnostic tool for observing dusty plasma state in laboratory plasmas.

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