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TOPOLOGY OF 3-DIMENSIONAL MANIFOLDS

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Abstract. This is the second paper from a trio which reviews some of the progress in low dimensional topology in the past century. Starting with the work of Poincaré in the last years of 19th century and first few years of 20th century, we review the major steps in putting 3-manifolds and their algebraic topology in a solid mathematical framework, and the important theorems which strengthened the understanding of 3-manifolds, including the prime decomposition theorem and JSJ decomposition of 3-manifolds. Highlighting the significance of hyperbolic 3-manifolds, proving the monster theorem and formulating the geometrization conjecture by Thurston has been a turning point in 3-manifold topology. The proof of geometrization conjecture by Perelman, using Ricci flow of Hamilton, affirmed that the fundamental group is an almost perfect invariant of closed 3-manifolds. Yet, it is not clear how geometric properties are reflected in the fundamental group, and its is difficult to verify whether two group presentations give isomorphic fundamental groups or not. Alternative approaches to the study of 3-manifolds and 4-dimensional cobordisms between them using abelian groups include, in particular, the theories which are formulated as topological quantum field theories (TQFTs). These approaches are also reviewed in the paper. In particular, a theorem of the author which addresses the strength of the later invariants in distinguishing 3-manifolds from the standard 3-sphere is discussed.

1. Introduction

The current paper focuses on the development of 3-manifold topology, which was basically initiated by celebrated papers of Poincaré in the last few years of 19th century [39, 40, 41, 42, 43, 44]. The manifolds studied by Poincaré were all smooth closed submanifold of the Euclidean space. The formulation of manifolds as abstract topological spaces lasted for another couple of decades. In fact, agreement on the

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"correct" definition of a manifold did not happen until after Whitney proved his embedding theorems [53]. The work of Poincaré initiated what is called "algebraic topology" today. Nevertheless, the construction of a rigorous language and the solid infrastructure for algebraic topology needed more time and happened in the first half of 20th century.

Poincaré conjecture motivated some of the most important developments of low dimensional topology in 20th century. The other critical development in 20th century was the study of hyperbolic 3-manifolds and the formulation of the "geometrization conjecture" by Thurston. The proof of the latter conjecture by Perelman in the beginning of 21st century [36, 38, 37] gave a very useful and effective understanding of closed 3-manifolds. We review the major steps in the aforementioned development. Moreover, a section is devoted to the progress in low dimensional topology which resulted from the introduction of gauge theory and topological quantum fields theories. In particular, one of the results of the author, which links the gauge theoretic invariants to the geometrization of 3-manifolds is discussed in the last section of the paper.

2. Main Results

The study of smooth manifolds by means belonging to "algebraic topology" was initiated by Poincaré, who published a series of 6 papers between 1899 and 1904, under the title "anaysis situs". He introduced the homology groups and the fundamental groups as tools for distinguishing manifolds from one another, and formulated what we know as Poincaré duality. The formulation of the following conjecture was perhaps the most motivating part of his papers:

Conjecture 2.1. *Suppose that M is a closed, connected and oriented 3-manifold with trivial fundamental group. Then there is a homeomorphism from M to S^3.*

In 1911 and 1912, Herman Weyl suggested a definition of an abstract manifold, while the equivalence of his definition and the definition used by Poincaré was not proved until the middle of 20th century. Two theorems are crucial for this equivalence. One of them is due to Moise [25] (proved in 1952) and the other one is due to Whitney (proved in 1936) [53].

Theorem 2.2 (Whitney embedding)**.** *For every positive integer m and every smooth Hausdorff m-manifold M which is of second category, there is an embedding of M in the Euclidean space $\mathbb{R}^{2m+1}$. Moreover, every continuous map $f : M \to N$ from a smooth m-dimensional manifold M to a smooth n-dimensional manifold N may be approximated by arbitrarily close smooth embeddings if $n > 2m$.*

Theorem 2.3 (Moise)**.** *Let M be a topological 3-manifold. Then M admits a maximal smooth atlas, giving a smooth structure on M. Moreover, every two such maximal smooth atlases are equivalent.*

A number of historic errors in the proof of some intuitively clear statements persuaded mathematicians that a rigorous language is needed for working with manifolds. Dehn lemma is a good example. It was first stated and proved (with a gap) by Dehn in 1910 [1]. The gap was noticed by Kneser in 1929 [18],

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and a complete proof was not given until 1957, when Papakyriakopoulos (Papa) proved the following stronger version [34, 35]:

Theorem 2.4 (loop theorem)**.** *Suppose that M is a compact 3-manifold with boundary and that $\gamma : S^1 \to M$ be a loop which is embedded on its boundary which represents a non-trivial element of $\pi_1(\partial M)$, while its image in $\pi_1(M)$ is trivial. Then, there is a proper embedding*

$$f : (\mathbb{D}, \partial\mathbb{D}) \to (M, \partial M)$$

from the standard 2-disk $\mathbb{D}$ to M, such that the image of $\partial\mathbb{D}$ in $\pi_1(\partial M)$ is non-trivial.

The proof of Papa also implied the “sphere theorem”. Kneser took the first step in this direction by proving the following theorem in 1929 [18]:

Theorem 2.5 (Kneser)**.** *Let M be a closed 3-manifold. Then, there is a finite sequence*

$$M = M_1, \dots, M_n = M'$$

of closed (possibly not connected) 3-manifolds such that M_{i+1} is obtained from M_i by reducing along a 2-sphere, while every connected component of M' is irreducible.

The proof of Kneser theorem also implies the “prime decomposition theorem” for closed 3-manifolds:

Theorem 2.6 (prime decomposition theorem for 3-manifolds)**.** *Let M be a closed oriented 3-manifold which is not homeomorphic to S^3. Then, there are non-trivial prime 3-manifolds $M_1, \dots, M_n$ such that M is homeomorphic to $M_1 \# \dots \# M_n$. Moreover, if $M \simeq M'_1 \# \dots \# M'_m$ is another prime decomposition of M, then $m = n$ and there is a permutation $\sigma \in S_n$ such that $M_i \simeq M'_{\sigma(i)}$.*

Moreover, a conjecture of Kneser was eventually proved by Stallings in 1959 [49, 50]:

Theorem 2.7 (prime decomposition theorem for 3-manifolds)**.** *Let M be a closed connected oriented 3-manifold and assume that*

$$\pi_1(M) = G_1 \star G_2 \star \dots \star G_n$$

where $G_1, \dots, G_n$ are non-trivial groups. Then there are closed 3-manifolds $M_1, \dots, M_n$ such that $\pi_1(M_i)$ is isomorphic to G_i and $M \simeq M_1 \# \dots \# M_n$.

Moreover, the importance of Seifert fibered 3-manifolds, which were introduced by Seifert in his PhD thesis, was noticed by several mathematicians. In particular, the following theorem was independently proved by Jaco-Shalen [15] and Johanson [17]:

Theorem 2.8 (JSJ decomposition)**.** *Let M be a closed irreducible 3-manifold. Then there is a minimal set of incompressible tori in M such that every connected component of their complement is either atoroidal or Seifert fibered. Moreover, this minimal set of incompressible tori is unique up to isotopy.*

A turning point in 3-manifold topology was the work of Thurston which highlighted the importance of hyperbolic manifolds. The following theorem of Mostow implies that hyperbolic structure in dimension 3, if it exists, is determined by the topology:

Theorem 2.9 (Mostow rigidity)**.** *Let* (M_1, g_1) *and* (M_2, g_2) *be two complete hyperbolic* n*-manifolds, with* $n > 2$*, and* $\imath : \pi_1(M_1) \to \pi_1(M_2)$ *be an isomorphism. Then there is an isometry* $f : (M_1, g_1) \to (M_2, g_2)$ *such that* $f_* : \pi_1(M_1) \to \pi_1(M_2)$ *gives the isomorphism* $\imath$*.*

Despite the initial misconception that hyperbolic manifolds are "rare", Thurston gave significant evidence that they are quite "common" [51]:

Theorem 2.10 (Thurston hyperbolic surgery)**.** *Let* M *be a hyperbolic manifold with toroidal boundary and finite volume. Then only finitely many Dehn fillings of* M *are non-hyperbolic.*

More importantly, Thurston proved the following theorem in 1982:

Theorem 2.11 (Thurston monster theorem)**.** *Let* M *be a compact irreducible atoroidal Haken* 3*-manifold such that* $\chi(\partial M) = 0$*. Then the interior of* M *may be equipped with a complete hyperbolic metric, so that the volume of* M *with respect to this volume form is finite.*

Thurston also formulated his "geometrization conjecture" which provides a very effective understanding of 3-manifolds. His conjecture, which implies the Poincaré conjecture as a very special case, remained open for more than 20 more years, until Perelman provided the first proof [36, 38, 37]:

Theorem 2.12 (Thurston geometrization conjecture-Perelman theorem)**.** *Let* M *be an irreducible closed* 3*-manifold and* $M_1, \ldots M_n$ *be the components of its JSJ decomposition. Then every* M_i *is either Seifert fibered or hyperbolic (with finite volume).*

As a corollary of the above theorem, the following may be proved:

Theorem 2.13. *Let* M_1 *and* M_2 *be closed irreducible* 3*-manifolds such that* $\pi_1(M)$ *and* $\pi_1(M_2)$ *are isomorphic. Then either* M_1 *and* M_2 *are homeomorphic, or they are both lens spaces.*

3. Conclusions

As the progress of 3-manifold topology indicates, and in particular Theorem 2.13 and Theorem 2.7 imply, the fundamental group of a 3-manifold is a very strong invariant which distinguishes many three-manifolds from each other. Nevertheless, it is usually a very hard problem to decide whether two presentations give isomorphic groups or not. Moreover, it is not clear how different geometric and topological properties are reflected in the fundamental group. Therefore, the resolution of geometrization conjecture did not close the study of low dimensional manifolds. Gauge theory has probably been the source of the most powerful alternative approach. It was initiated by the work of Floer [7, 8, 11, 10, 9, 12], which paved the way for the introduction of topological quantum field theories (TQFTs), and the

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introduction of instanton Floer theory and Seiberg-Witten theory. Ozsváth and Szabó introduced a similar theory in 2001 [32, 31, 33]. Kutluhan, Lee and Taubes proved that instanton Floer homology and Ozsváth-Szabó Floer homology (also called Heegaard-Floer homology) are equivalent [20, 21, 22, 23, 24].

We may thus turn our attention to Ozsváth-Szabó invariants. The theory associates the groups $\widehat{\mathrm{HF}}(M), \mathrm{HF}^+(M), \mathrm{HF}^-(M), \mathrm{HF}^\infty(M)$ to a 3-manifold M and the maps

$$\Phi_W^\bullet : \mathrm{HF}^\bullet(M_1) \to \mathrm{HF}^\bullet(M_2), \quad \forall \bullet \in \{+, -, \infty, \wedge\}$$

for every cobordism W from M_1 to M_2. The groups and the maps respect the corresponding decompositions according to Spin^c structures. They reflect many interesting properties, and several applications of these invariants are mentioned in the last section of the paper.

The hat theory (i.e. the groups $\widehat{\mathrm{HF}}(M)$) are the most convenient versions of the theory. It is an abelian group which may be computed combinatorially (i.e. using a computer). Yet, it is conjectured that it is relatively powerful in distinguishing 3-manifolds from S^3:

Conjecture 3.1 (Ozsváth-Szabó)**.** *Let M be a prime closed 3-manifold and assume that $\widehat{\mathrm{HF}}(M) = \widehat{\mathrm{HF}}(S^3) = \mathbb{Z}$. Then M is homeomorphic to either S^3 or the Poincaré homology sphere P.*

The best result in the direction of Conjecture 3.1 is the following theorem of author [6], which is proved based on [5] and [4]:

Theorem 3.2. *Let M be a prime closed 3-manifold and assume that $\widehat{\mathrm{HF}}(M) = \widehat{\mathrm{HF}}(S^3) = \mathbb{Z}$. Then M is homeomorphic to either S^3 or the Poincaré homology sphere P, or M is hyperbolic.*

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