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BOLLOBÁS LEMMA: ANALYSIS, APPLICATION, ALGORITHM

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Abstract. Many of the theorems that were developed by researchers in the 1960s and 1970s in combinations and graph theory as mathematical theorems, are still used today in designing algorithms and solving new problems. The purpose of this article is to show that useful things happened in the past that we should not forget and use them with a creative eye. Bollobás Lemma [1], which was proposed in the 1970s, is a well-known problem in combinatorics. In this problem, we have a family of sets $A_1, A_2, \ldots, A_m$ each with size a and another family of sets $B_1, B_2, \ldots, B_m$ each with size b. The goal is to find the maximum size m, the number of sets so that for each index i we have $A_i \cap B_i = \emptyset$ and also $A_i \cap B_j \neq \emptyset$ where $(i \neq j)$. The Bollobás lemma expresses the upper bound for the maximum number of these sets as $m \leq \binom{a+b}{b}$. In this paper, after stating the versions of the lemma and the existing proof for this lemma, we present another probability-based proof for the Bollobás Lemma, and then with a different look at this combinatorial problem, we investigate the interesting applications of this lemma in graph theory problems and parameterized algorithms.

1. Introduction

Suppose we have 20 astronauts and we want to choose a group of 6 of them to send to space, but we know that 3 of them are going to get sick the day before the flight, and the group that has a sick person cannot go to space. The question that arises here is how many groups should be formed to be sure that at least one of them can go to space? An obvious answer to this problem is $\binom{20}{3}$, but it is

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clear this value grows factorially according to the number of primary astronauts. So we have to look for a way to calculate the answer to this problem with less calculations. In the following, we introduce the Bollobás lemma, which can be used to reduce the amount of calculations for this problem to $\binom{9}{3}$.

2. Main Results

Lemma 2.1. [1] *If $A_1, A_2, \dots, A_m$ be minimal family of sets of sizes $a_1, a_2, \dots, a_m$ and so on $B_1, B_2, \dots, B_m$ be another family of sets with sizes $b_1, b_2, \dots, b_m$ such that $A_i \cap B_i = \emptyset$ for each i and $A_i \cap B_j \neq \emptyset$ for any $i \neq j$, then*

$$\sum_{i=1}^{m} \binom{a_i + b_i}{b_i}^{-1} \leq 1. \tag{2.1}$$

We present two proofs for this lemma, the existing proof for this lemma in [1], and a new probability-based proof. Then with a different look, we investigate the interesting applications of this lemma in graph theory problems and parameterized algorithms.

Lemma 2.2. *[General Case] If $A_1, A_2, \dots, A_m$ form a minimal family of sets with sizes $a_1, a_2, \dots, a_m$ respectively, and similarly $B_1, B_2, \dots, B_m$ form another family of sets with sizes $b_1, b_2, \dots, b_m$ such that $A_i \cap B_i = \emptyset$ and $A_i \cap B_j \neq \emptyset$ $(i \neq j)$, then*

$$\sum_{i=1}^{m} \binom{a_i + b_i}{b_i}^{-1} \leq 1. \tag{2.2}$$

It is clear that by substituting a and b for the parameters a_i and b_i respectively, Equation (2.1) can be reached.

$$\sum_{i=1}^{m} \binom{a + b}{b}^{-1} = m\binom{a + b}{b}^{-1} \leq 1 \Longrightarrow m \leq \binom{a + b}{b} \tag{2.3}$$

Lemma 2.2 provides an upper bound for the number of members in the family. Now we present two proofs for this lemma in the general case.

Proof 1.

Definition 2.3 (Separating Permutation)**.** *Let A and B be two subsets of the set $\{x_1, \dots, x_n\}$ such that $A \cap B = \emptyset$. A permutation $(x_{i_1}, \dots, x_{i_n})$ is a separating permutation of (A, B) if all elements in the set A precede those in the set B, i.e., for every $x_k \in A$ and $x_l \in B$, $x_k < x_l$.*

Claim: Each permutation separates at most one pair (A, B).

Proof: Suppose to the contrary that there exists a permutation that separates two pairs (A_i, B_i) and (A_j, B_j). Also, according to the definition, we know that $A_i \cap B_j \neq \emptyset$ and $A_j \cap B_i \neq \emptyset$. Now two cases arise: either the smallest element of B_j is smaller than the smallest element of B_i, or vice versa. Here

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we assume the first case occurs (Figure 1). But this is impossible since we know that A_j and B_i have common elements, leading to a contradiction, as two pairs cannot be separated by one permutation.

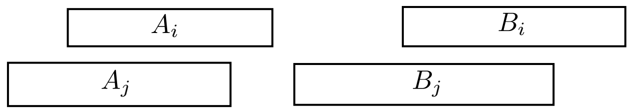

FIGURE 1. Two subsets (A_i, B_i) and (A_j, B_j)

Question: How many permutations can separate the pair (A, B)?
Answer: If $a = |A|$ and $b = |B|$, then the number of ways we can select sets A and B is $\binom{n}{a+b}$. The number of permutations of A and B is $a!b!$. The number of permutations for the remaining elements not chosen is $(n-a-b)!$. Hence, we have:

$$\binom{n}{a+b}.a!b!.(n-a-b)! = n!\binom{a+b}{a}^{-1}. \tag{2.4}$$

Now, if we calculate the number of permutations for each pair (A, B), we get the following:

$$\sum_{i=1}^{m} n!\binom{a_i+b_i}{a_i}^{-1}, \tag{2.5}$$

which is always less than or equal to $n!$, the total number of permutations over n. Thus, the following inequality is always valid:

$$\sum_{i=1}^{m} n!\binom{a_i+b_i}{a_i}^{-1} \leq n! \implies \sum_{i=1}^{m} \binom{a_i+b_i}{b_i}^{-1} \leq 1. \tag{2.6}$$

□

Proof 2. We give a probabilistic proof. Assume π is a random permutation of the set $\{x_1, x_2, \ldots, x_n\}$. In the previous proof, it was shown that the number of permutations that separate A and B is $n!\binom{a+b}{a}^{-1}$, i.e.,

$$P(\ \pi \ separates\ A\ and\ B\ ) = \binom{a+b}{a}^{-1}. \tag{2.7}$$

Additionally, it was proved that the two probabilities

- π separates A_i and B_j,
- π' separates A_j and B_j,

are independent since each permutation only separates one pair. Hence, the sum of probabilities for all m pairs can be written as $\sum_{i=1}^{m} \binom{a_i+b_i}{b_i}^{-1}$, and since it is a probability, it always has a value less than or equal to 1, i.e.,

$$\sum_{i=1}^{m} \binom{a_i+b_i}{b_i}^{-1} \leq 1. \tag{2.8}$$

□

It is easily seen that we can use the Bollobás Lemma to solve the astronaut selection problem, since all conditions for using this lemma hold, i.e., the astronaut groups we select and the group of people who get sick should have no intersection.

2.1. **Some Application of Bollobás Lemma.** In this section, we will examine some applications of this lemma in proving and solving various problems such as:

- A family of sets F is antichain if for every $A, B \in F$ with $A \neq B$, we have $A \nsubseteq B$.
  If F be a antichain on the set X with n members then:
$$\sum_{A \in F} \binom{n}{|A|}^{-1} \leq 1.$$
- The minimum set S of the vertices of the graph G is called vertex cover if S contains at least one vertex of each edge of the graph. The graph G is called k-edge-critical if G has a vertex cover of size k and removing each edge e leads to a vertex cover with size $k-1$. Every k-edge-critical graph contains at most $\binom{k+1}{2}$ edges.
- $S \subseteq V(G)$ is an independent set if there are no edges between the vertices in S. There is no n vertex graph that has more than $3^{n/3}$ maximum independent set.
- A graph with n vertices has at most $O(1.6181^n)$ minimal separator.

3. Summary of Proofs/Conclusions

In this article, after considering the states of the lemma and the existing proof for this lemma, we provide another probability-based proof for the Bollobás lemma. Then we examine the interesting applications of this lemma in problems such as, antichain sets [3], Edge k-critical graphs, minimal separator [4] and representative sets [7].

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