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KNOT THEORY, FROM PAST TO PRESENT

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Abstract. In this paper, which is the first paper from a trio on important developments of low dimensional topology in the past 100 years, we review the history and major developments in knot theory. This historic account includes the initial attempts at formulating some of the main questions about knots in a mathematical language, putting the definitions and arguments in a rigorous mathematical framework, and employing tools from other fields of mathematics to extract interesting and intriguing results in knot theory. The study starts from Tait's work in nineteenth century and reviews the important steps taken before the introduction of gauge theory. In particular, we will review the prime decomposition of knots and various polynomial invariants constructed for knots and links. We finish the paper by discussing some of the important conjectures in knot theory which have surprisingly simple statement. We also review some of the recent developments around the aforementioned conjecture, including some theorems including contributions from the author.

1. Introduction

This is the first paper from a trio (including the present paper, [6] and [5]) which described some of the major developments in low dimensional topology in the past 100 years.

Among topological objects in low dimensions, knots are distinguished as objects with extreme intuitive simplicity as well as significant topological complexity. Knots have a long history in different civilizations [3, 8]. Nevertheless, the first mathematical/scientific attempts at understanding knots go

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back to 18th century, when Sir William Thomson (Lord Kelvin) used knots in his model for atoms[14]. Later, Mathematicians got interested in the study of knots as well. In particular, Gauss introduced the notion of linking number in 19th century. In 1867, Tate introduced the first table of knots, and ordered the knots according to their crossing numbers. He also presented the first complete list of alternating knots with at most 10 crossings [10, 14].

The work of Tate is distinguished from other approaches of his time, and some of the concepts and questions he introduced remained in the center of attention for more than a century. In particular, his conjectures (which were proved in the second half of 20th century) are of extreme importance [10].

In last years of 19th century and the first few years of 20th century, algebraic topology came to birth in the work of Poincaré, and in particular, the concept of fundamental group was introduced. The fundamental group of the knot complement later proved to be a very strong knot invariant. Among other developments in the beginning of 20th century , which influenced the progress of knot theory are the representations of the braid groups and the introduction of a polynomial invariants by Alexander. Moreover, Reidemeister's theorem on how different diagrams of a knot are changed to one another, paved the way for a more rigorous and reliable approach to constructing knot invariants.

Classical methods were used in some of the main developments in the later years of the 20th century. In 1954, the prime decomposition theorem for knots was proved, which is probably one of the most important conceptual theorems in knot theory. Jones polynomial and HOMFLY polynomial were constructed in 1984. Moreover, the progress in low dimensional topology resulted in tools and developments in knot theory. This line of progress started with the influence of gauge theory on knot theory.However, in this paper we focus on the era before the introduction of gauge theory.

In the last section of the paper, some of the major conjectures and open questions in knot theory are reviewed. We have selected the conjectures and open questions which have simple statements, and are thus accessible for a wide range of audience.

2. Main Results

The main theorems and developments covered in this paper include the following major results. We start with Tate conjectures, which were formulated towards the end of 19th century and motivated several later developments:

Conjecture 2.1 (Tate Conjectures)**.** *Suppose that L is an alternating link. Then the following are true.*

- *The number of crossings in every alternating diagram of L which does not have any nugatory crossings is precisely the crossing number $c(L)$ of L.*

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- *The wirth numbers of every two alternating diagrams of L which do not have any nugatory crossings are the same.*
- *Every two alternating diagrams of L which do not include nugatory crossings are changed to one another by a finite sequence of flypes.*

Schubert presented an important contribution in 1954, which is called the Prime Decomposition of knots [13].

Theorem 2.2 (Prime Decomposition Theorem for Knots)**.** *Every nontrivial knot K in $\mathbb{R}^3$ may be written as the connected sum of prime knots. Moreover, if*

$$K = K_1\#\cdots\#K_n = K_1'\#\cdots\#K_m'$$

are two decompositions of K to (nontrivial) prime knots, then $m = n$ and there is a permutation $\sigma \in S_n$ such that K_i' and $K_{\sigma(i)}$ are equivalent.

In the beginning of 20th century, the relation between knot theory and the study of braid groups was an important line of research. In particular, the following theorem of Alexander from 1923 is considered an important step:

Theorem 2.3 (Alexander)**.** *For every link L, there is a positive integer n and a braid $b \in B_n$ so that the closure of b gives L.*

The contribution of Alexander did not stop here. In 1928, he introduced the first polynomial invariants for knots [1]. His construction used the work of Reidemeister [12] and the previous work of Artin on braid groups. To every diagram D of a knot K, he associated a $n \times (n+2)$ matrix $A_D'(t)$ with monomial entries, called the Alexander matrix. By removing the last two columns of A_D', he obtains a square matrix $A_D(t)$, and defined

$$\Delta_K(t) \doteq \det(A_D(t)).$$

Here $\doteq$ indicates that the equality is modulo multiplication by a factor $\pm t^k$ for some integer k.

Theorem 2.4. *The polynomial $\Delta_K(t)$ is an invariant of the knot K, up to multiplication by a monomial $\pm t^k$, and does not depend on the choice of the diagram D.*

Conway normalized the Alexander polynomial, and got rid of the ambiguity (of multiplication by a factor $\pm t^k$) [4]. For other new knot polynomials, mathematicians waited more than 50 years. Jones introduced his polynomial in 1984 [9], while 4 months his construction was generalized to give the HOMFLY polynomial [7]. By a skein we mean three links K^0, K^+ and K^- which have diagrams that are identical outside a small disk, while inside that disk they look like Figure 14 of the paper. The construction of HOMFLY polynomial may be stated as the following theorem.

Theorem 2.5 (Existence of HOMFLY polynomial)**.** *We may associate a polynomial $P_K \in \mathbb{Z}[a, z]$ to every link K so that*

- *If K and K' are equivalent then $P_K = P_{K'}$;*
- *For the unknot O, we have $P_O = 1$;*
- *For every skein K^0, K^-, K^+ we have*

$$\frac{1}{a}P_{K^+} - aP_{K^-} = zP_{K^0}.$$

Although HOMFLY polynomial is capable of distinguishing many knots from one another, there are ways to change a knot K to a different knot K' which shares the same HOMFLY polynomial. In particular, mutation provides one such operation. For instance, Conway mutation changes the Conway knot to Kinoshita-Terasaka knot, as illustrated in Figure 17 of the paper. For a knot invariant, it is always an interesting question to verify if it distinguishes between mutant knots.

3. Conclusions

The paper concludes with some open questions and conjectures in knot theory, with different flavors. The open questions include the following question about the existence of an algorithm for distinguishing a knot from the unknot.

Question 3.1. *Is there a polynomial-time algorithm than can verify if a knot is non-trivial?*

We do not know of any nontrivial knots with trivial Jones polynomial. If the following conjecture is true, the above question may be answered positively.

Conjecture 3.2. *The Jones polynomial of a non-trivial knot is nontrivial*

The crossing number of knots is the source of many questions as well. Let us denote by $N(c)$ the number of non-equivalent knots with at most c crossings.

Question 3.3. *Does the limit*

$$\lim_{c\to\infty} \big(N(c)\big)^{\frac{1}{c}}$$

exist? If so, what is its value?

It seems that the crossing number behaves pretty well under connected sum. Nevertheless, proving the following natural conjecture is still wide open.

Conjecture 3.4 (Additivity of Crossing Number)**.** *For every two knots K and K' we have*

$$c(K_1\#K_2) = c(K_1) + c(K_2).$$

The best known result is the following theorem, which is due to Lackenby [11]:

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Theorem 3.5. *For every two knots K and K' we have*

$$\frac{c(K_1)+c(K_2}{152} \leq c(K_1\#K_2) \leq c(K_1)+c(K_2).$$

Let $u(K)$ denote the unknotting number of a knot K. The additivity of $u(K)$ under connected sum is also wide open.

Conjecture 3.6 (Additivity of Unknotting Number)**.** *For every two knots K and K' we have*

$$u(K_1\#K_2) = u(K_1)+u(K_2).$$

It is not even known whether for a pair of knots K_1 and K_2, the unknotting number of $K = K_1\#K_2$ is always at least a fixed multiple of $u(K_1)$. The following theorem of Alishahi and the author [2] is considered a significant progress in the direction of Conjecture 3.6.

Theorem 3.7. *For every knot K, there is a lower bound $\mathfrak{t}(K)$ for $u(K)$ so that for every two knots K and K' we have*

$$\mathfrak{K}_1\#\mathfrak{K}_2 \geq \max\{\mathfrak{K}_1, \mathfrak{K}_2\}.$$

Moreover, for the torus knot $K = T_{p,q}$ we have $\mathfrak{t}(K) = \min\{p,q\}-1$.

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