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MARIO PIERI'S AXIOMATIZATION OF GEOMETRY

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Abstract. In this paper, we discuss the method of the Italian mathematician, Mario Pieri, for axiomatization of geometry, which is based only on two undefined terms, point and motion. We examine the postulates proposed by him, definitions of various geometrical concepts and some important theorems arising from those postulates.

1. Introduction

After the conclusion of a long debate about the fifth postulate of Euclid known as the parallel postulate in the early 19th century and the emergence of non-Euclidean geometries, the attention of mathematicians was drawn to other postulates of Euclid and the question, whether or not, in general, the set of postulates presented by him is a satisfactory system of principles, was taken into consideration. The first person who tried to present an improved system of postulates for geometry was Moritz Pasch, a German mathematician who did this in 1882 in a book called Lectures in New Geometry [3]. From his point of view, some important geometrical concepts in Euclid's book, such as betweenness of points as well as the congruence of geometric shapes are not precisely defined, nor are principles established for the exact identification of these concepts. The undefined terms for Pasch at

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the beginning of the discussion were points and line segments, to which were later added the concepts of flat surface and congruence of geometrical shapes. From Pasch's point of view, line segment as an undefined term is preferable to line because it is more tangible, and by introducing line segment as an undefined term and establishing principles for it, it is also possible to define the concept of betweenness of points at the same time. Regarding congruence, Pasch got his idea from the concept of rigid motion, and according to him, congruence between two geometrical shapes intuitively means that one of the them can be moved in such a way that it fits properly on the second one. Of course, the problem in this regard is that geometrical shapes can not necessarily be moved independent of each other. For example, if two shapes have common points, with each movement of one, the other's common points also move. Of course, Pasch overcame this problem in this way that if there is a third shape that can be moved with respect to the above two shapes and fits both, then both of them will also be congruent. Of course, the concept of motion for Pasch was just an intuitive idea to build the concept of congruence and formulate the principles related to it, and in the ten postulates that Pasch established for congruence, no name of motion was mentioned.

The next person was Giuseppe Peano, an Italian mathematician, who in 1889, inspired by Pasch's work, wrote a booklet called Principles of Geometry [4]. The above work was almost a rewriting of Pasch's principles and theorems in logical language. Of course, the topics related to congruence are absent in this small treatise. Five years later, Peano expanded his previous article to a considerable extent in another article called the Basics of Geometry, which is more detailed, and added explanations to it [5]. But the most important difference between this and the previous article is the topics which are related to congruence. In this part, Peano, after referring to principles of congruence by Pasch, took another method and instead of congruence, he introduced motion itself, as an undefined term and established principles for it. Unlike Pasch, whose perception of the concept of motion was really a physical movement, Peano considered motion as a mapping that corresponds points with properties reflected in principles. After introducing motion and its principles, at the end of the article, Peano presented a relatively detailed discussion about specific types of motion, including reflection, translation, and rotation.

The next person was Mario Pieri, another Italian mathematician, who went even further and claimed that it is possible to define other geometrical concepts with only two undefined terms of point and motion. In 1900, almost simultaneously with Hilbert's Principles of Geometry, he wrote an essay called the Treatise on Point and Motion [6] and he used the above idea there and tried to prove all the theorems of the first three books of Euclid. Some of his basic principles regarding motion are similar to Peano, but he has followed a different approach. In this article, we examine the method used by Pieri and review the postulates introduced by him, concepts based on these postulates and the important theorems derived from them. For more information about Mario Pieri and his role in

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development of mathematics, refer to [1, 2]. Especially in [2], there is a translation of the Treatise on Point and Motion in English, which is also used in writing this article.

2. Main Results

Throughout this section, a, b, c are distinct points. Pieri in his monograph first introduces some postulates to explain the nature of points and motions. The first concept that he defines by means of motions is the collinearity of three points. He calls three points collinear if there exists a pure motion that keeps all three points fixed. Next, he defines the line joining two points a, b as the set of points which are collinear with them and defines a plane abc as the set of all points on any of the lines that connect a to points on the line bc or connect b to the points on the line ac or connect c to the points on the line ab. His most important tool for introducing the geometric concepts by motions is sphere. In his sense the sphere through b with center a (denoted by b_a) is the set of all points x, such that, there is a motion that takes b to x and keeps a fixed. Next he introduces some postulates about intersection of lines and spheres and defines reflection of b across a, as the point that lies on the line ab and on the sphere b_a but different from b. He mentions a postulate which guarantees the existence of a motion which carries a onto b and maps some point of ab to itself and calls that fixed point, the midpoint of a, b denoted by $a|b$. Next he defines a circle as a set of points that lie on a sphere and at the same time in a plane that contains the sphere's center. Then he says the pair (a, c) is perpendicular to the pair (a, b) (denoted by $(a, c) \perp (a, b)$), if there exists a motion that maps c into a point different from c but belonging to the line ca and keeps the points a, b fixed. He calls two figures congruent to each other, whenever there exists a motion so that the one figure is transformed precisely into the other. Next he defines the polar sphere of the points a, b as the sphere $a_{a|b} = b_{a|b}$ and says the point x is internal to a sphere b_a when there exist two distinct points on the sphere having x as midpoint. He says a point x is between a, b if a, b, x are collinear and x is internal to the polar sphere of a, b. Next he defines the segment $|ab|$ as the set of all points that lie between a and b, or that coincide with a or with b. By introducing the concept of betweennes, the rest of his approach is similar to standard axiomatic methods and he introduces the concepts of ray, triangle and halfplane similarly. He concludes his paper with a discussion of continuity principle.

3. Summary of Conclusions

In this article, we have briefly examined Mario Pieri's method for axiomatization of geometry by two undefined terms, point and motion. The concept of motion was first used by Pasch as an idea to introduce the undefined concept of congruence. Then Peano directly used motion as one of the undefined concepts and used it to define congruence. But Peiri's method was unique in that he considered point and motion as the only undefined terms and defined other geometric concepts by

them and according to what was discussed in this article , it can be concluded that his effort was quite successful.

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