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AN INVESTIGATION OF KURATOWSKIS DEFINITION OF AN ORDERED PAIR

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Abstract. In this paper, we study the definitions of ordered pair from the point of view of Kuratowski, Wiener, Hausdorff and Morse. We first present the possible definitions of an ordered pair, and then examine the structure of the Cartesian product of sets, relations, and functions. This paper can be useful for mathematical researchers in creating new insight and better understanding of the concept of ordered pair and its related mathematical concepts, such as ordered and unordered Cartesian product, relation and function, etc.

1. Introduction

The aim of this paper is the study of the ordered pair, especially the examination of the possible definitions for the ordered pair. Every theory in mathematics has axioms, definitions, symbols and theorems. In the theory of sets, the ordered pair is considered as a definition. This concept also has a natural meaning in the physical world. In 1914, Wiener gave the first simple definition of an ordered pair. Later, Kazimierz Koratowski in 1921 suggested a simpler definition of ordered pair than Wiener in set theory, so that relation and function were easily defined. Zermelo's axioms of set theory of 1908 provide a theory which can found all of classical mathematics. The paper is organized as follows.

Keywords: Kuratowski's Definition of an Ordered Pair, Cartesian produc.

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In section 1, we collect the basics concepts and present the history of the ordered pair. In section 2, we provide a short summary of the first-order structure and logic concepts. We recall that the language desired in this paper is the relation language. In section 3, we study some of the cases of the definition of ordered pair. In section 4, we discuss possible definitions for the ordered pair that it is not satisfiable in formula $(a, b) = (c, d) \Longleftrightarrow a = c,\ b = d$. The end section some results are set out.

2. Main Results

The obvious approach is to translate or codify naive set theory into a formal first order theory. By a language we mean a mapping σ, the domain of which is any set and the range of which is a set of integers. By a σ-symbol we mean an element of the domain of σ. A σ-symbol s is said to be an operation symbol if $\sigma(s) \geq 0$; it is said to be a relation symbol if $\sigma(s) < 0$. A first-order language is a set

$$\mathcal{L} = \mathcal{F} \dot{\bigcup} \mathcal{R}$$

where

$$\mathcal{F} = \dot{\bigcup_{n \geqslant 0}} \mathcal{F}_n, \quad \mathcal{R} = \dot{\bigcup_{n \geqslant 1}} \mathcal{R}_n.$$

Notice that objects $\mathcal{F}_n$ operation symbols n-ary and objects $\mathcal{R}_n$ relation symbols n-ary and objects $\mathcal{F}_0$ constant symbols are said. The language is called algebraic language if $\mathcal{R} = \emptyset$. Also, the language is called relational language if $\mathcal{F} = \emptyset$. By a relation n-ary relation on a set A we mean a subset of A^n. We denote by

$$A = (A, f^A, R^A, f \in \mathcal{F}, R \in \mathcal{R}) \qquad .$$

Note that the standard logical symbols of a formal first order (relation) language, namely, are

$$\neg, \quad \exists, \quad \neg \quad \vee, \quad =, \quad (,)$$

Here, in this paper, we introduce a new function symbol of arity 2 , J, by

$$J(x, y) = \{\{x\}, \{x, y\}\}$$

It is customary to denote the term $J(x, y)$ by (x, y).

2.1. **Definition Unordered Cartesian Products of Arbitrary Family of Sets and Unordered $\triangle$-tuples:** Suppose $A = \{A_x \,|\, x \in \triangle\}$ is a family of sets. We define the unordered Cartesian product $\prod_{x\in\triangle} A_x = \prod\{A_x \,|\, x \in \triangle\}$ as the set of all functions

$$f : \triangle \longrightarrow \cup_{x\in\triangle} A_x$$

satisfying $f(x) \in A_x$ for each x in $\triangle$.

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2.2. **Definition Ordered Cartesian Products of Arbitrary Families of Sets:** Suppose $A = \{A_x \,|\, x \in \triangle\}$ is a family of sets and $(\triangle, \leq)$ is partially order set. We define the ordered Cartesian product $(\prod_{x\in\triangle} A_x, \leq) = \prod\{A_x \,|\, x \in \triangle\}$ as the set of all functions $f : \triangle \longrightarrow \cup_{x\in\triangle} A_x$ satisfying $f(x) \in A_x$ for each x in $\triangle$.

Recall that many Mathematicians tried to define ordered pair. We are ready to list theorms that we proved:

Theorem 2.1. $(a, b) = \{\{\{a\}, \emptyset\}, \{\{b\}\}\}$ *if and only if* $(a, b) = (c, d) \Longleftrightarrow a = c,\ b = d$

Theorem 2.2. $(a, b) = \{\{a, \emptyset\}, \{b, \{\emptyset\}\}\}$ *if and only if* $.((a, b) = (c, d) \Longleftrightarrow a = c,\ b = d)$

Theorem 2.3. $(a, b) = \{a, \{a, b\}\}$ *if and only if* $.(a, b) = (c, d) \Longleftrightarrow (a = c,\ b = d)$

Theorem 2.4. $(a, b) = \{\{a\}, \{\{a\}, b\}\}$ *if and only if* $.(a, b) = (c, d) \Longleftrightarrow (a = c,\ b = d)$

It is not hard to show that the following definitions for the ordered pair are not satisfiable formulas:

$$(a, b) = (c, d) \Longleftrightarrow a = c,\ b = d$$

-$(a, b) = \{\{b\}, \{a, b\}\}$
-$(a, b) = \{\{a, b\}\}$
-$(a, b) = \{a, b\}$
-$(a, b) = \{a\}$
-$(a, b) = \{b\}$

3. Conclusions

We conclude that the Kuratowski's definition of the ordered pair is consistent with finding coordinates in Euclidean spaces and other space coordinates.

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