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Journal of Mathematics and Society
ISSN (print): 2345-6493, ISSN (on-line): 2345-6507
Vol. 9 No. 1 (2024), pp. 33-50.

http://www.ui.ac.ir

SHAPE OPERATOR OF AN $(N-1)$-DIMENSIONAL DISTRIBUTION ON AN N-DIMENSIONAL MANIFOLD AND THEIR CLASSIFICATION

MEHRAN AMINIAN* AND MEHRAN NAMJOO

Abstract. This paper aims to study of shape operator of an $(n-1)$-dimensional distribution on an n-dimensional smooth manifold. In this study firstly we state formulae for the shape operator and its symmetric and anti-symmetric components and in continuation we show their relationships with some notions such as integrability, totally umbilic and totally geodesic. Finally, by considering at most two eigenvalues for the shape operator, we classify this distribution and their foliations in simply connected space forms.

1. Introduction

The study of surfaces in ordinary three-dimensional space was expanded by Gauss in the early 19th century by introducing the concepts of first and second fundamental forms and curvature. This approach was generalized by studying the submanifolds of a Riemannian manifold. Details of this matter in [10, 7, 11] have been studied. But other aspects need to be studied and in this paper, we intend to examine them.

In differential geometry, a distribution on a manifold is an association of vector subspaces that have special properties, and often this distribution is a subbundle of a tangent bundle. The distributions that have the integrability condition create a foliation on the manifold, that is, they separate the

Keywords: Distribution, shape Operator, second fundamental form.

Communicated by Mohamad Reza Pouryayevali.

Article Type: Research Paper.

*Corresponding author.

Received: 2023-08-21, Accepted: 2023-12-31, Published Online: 23-04-2024.

Cite this article: M. Aminian and M. Namjoo, Shape operator of an $(n-1)$-dimensional distribution on an n-dimensional manifold and their classification, *Journal of Mathematics and Society*, **9** no. 1 (2024) 33–50.

http://dx.doi.org/10.22108/msci.2023.138841.1600 .

manifold into smaller submanifolds. These concepts have many applications in different fields of mathematics, such as integrable systems, Poisson geometry, differential topology, etc [17, 2, 6].

The study of the geometry of regular distributions is a natural extension of the study of the geometry of submanifolds. The mode of integrable distributions is in accordance with the study of foliations. For more details we refer the reader to [16].

If the distribution is not integrable, then the tensor field of its shape operator, it is not symmetrical and its decomposition into symmetric and antisymmetric parts is related to the geometric properties of the distribution. The second fundamental form as well as the shape operator, is the main and basic tool which this research aims to study. We prove different properties of symmetric and antisymmetric components. Finally, by considering at most two eigenvalues for the shape operator, we classify the distribution and its foliations in simply connected space forms.

2. Main Results

Suppose (M^n, g) be a Riemannian manifold, $\mathcal{H}$ an $(n-1)$-dimensional regular distribution (horizontal distribution) on M and $\mathcal{V}$ the distribution of its orthogonal complement to the Riemannian metric g (vertical distribution). So $TM = \mathcal{H} \oplus \mathcal{V}$. Let H and V be smooth $(1,1)$-tensor fields that attribute to a vector field (to a vector) its horizontal and vertical parts, i.e. $H(E) = E - \langle E, N \rangle N$ and $V(E) = \langle E, N \rangle N$, where N is an unit vector field may be locally defined and it is perpendicular to the horizontal distribution. Here the symbol $\langle , \rangle$ stands for the inner multiplication of the metric g, and the second fundamental form of $\mathcal{H}$ is defined in the following form

$$B^{\mathcal{H}}(E, F) = V(\nabla_{H(E)} H(F)),$$

where ∇ denotes the Levi-Civita connection of metric g and $E, F \in \mathcal{X}(M)$.

Definition 2.1. [12] *Suppose N be an unit vertical vector field. The shape operator S obtained from N, is given by the following relation*

$$\langle SX, Y \rangle = \left\langle B^{\mathcal{H}}(X, Y), N \right\rangle,$$

where X, and Y, are horizontal vector fields.

The following proposition is similar to that in [12].

Proposition 2.2. *Shape operator specifies a linear operator $S : \mathcal{H}_p \to \mathcal{H}_p$, at any point $p \in M$ and for each $v \in \mathcal{H}_p$, $Sv = -\nabla_v N$.*

Remark 2.3. *If the vertical vector field N (which may be locally defined) is replaced with $-N$, then the sign of S changes. So even if M lacks a vector field that is vertical throughout, then the shape operator S is defined globally up to sign. The sign of ambiguity should be removed from the inherent formulas.*

http://dx.doi.org/10.22108/msci.2023.138841.1600

Symmetric and antisymmetric components of $B^{\mathcal{H}}$ which is denoted by $B_s^{\mathcal{H}}$ and $B_a^{\mathcal{H}}$, respectively are defined as follows:

$$B_s^{\mathcal{H}}(E,F) = \frac{1}{2}\left(B^{\mathcal{H}}(E,F) + B^{\mathcal{H}}(F,E)\right),$$
$$B_a^{\mathcal{H}}(E,F) = \frac{1}{2}\left(B^{\mathcal{H}}(E,F) - B^{\mathcal{H}}(F,E)\right),$$

where $E, F \in \mathcal{X}(M)$. Therefore $B^{\mathcal{H}} = B_s^{\mathcal{H}} + B_a^{\mathcal{H}}$. Motivated by this, we offer the following definition.

Definition 2.4. *Symmetric and antisymmetric components of S, which is denoted by S_s and S_a, respectively are defined by*

$$\langle S_s X, Y\rangle = \left\langle B_s^{\mathcal{H}}(X,Y), N\right\rangle,$$
$$\langle S_a X, Y\rangle = \left\langle B_a^{\mathcal{H}}(X,Y), N\right\rangle,$$

where X and Y, are horizontal vector fields. As a result $S = S_s + S_a$, where

$$S_s = \frac{1}{2}\left(S + S^t\right),$$
$$S_a = \frac{1}{2}\left(S - S^t\right),$$

and S^t denotes the transpose of S.

Proposition 2.5. *For each $v \in \mathcal{H}_p$, the following relationships are established:*

$$\begin{aligned} S_s v &= -\frac{1}{2}\left(\nabla_v N + H((\nabla N)^t v)\right) \\ &= -\frac{1}{2}\left(\nabla_v N + (\nabla N)^t v - \langle v, \nabla_N N\rangle N\right), \end{aligned}$$

and

$$\begin{aligned} S_a v &= -\frac{1}{2}\left(\nabla_v N - H((\nabla N)^t v)\right) \\ &= -\frac{1}{2}\left(\nabla_v N - (\nabla N)^t v + \langle v, \nabla_N N\rangle N\right). \end{aligned}$$

Similar to what was stated in [3, 12], we arrive at the following proposition.

Proposition 2.6. *For any horizontal vector field X, Y, we have*

$$\begin{aligned} \langle S_s X, Y\rangle &= \left\langle B_s^{\mathcal{H}}(X,Y), N\right\rangle \\ &= -\frac{1}{2}(\mathcal{L}_N g)(X,Y). \end{aligned}$$

Definition 2.7. *We call $\mu^{\mathcal{H}}$ mean curvature and it is defined in the following form*

$$\begin{aligned}\mu^{\mathcal{H}} =& \frac{1}{n-1}\mathrm{tr}B^{\mathcal{H}}\\ =& \frac{1}{n-1}\mathrm{tr}B^{\mathcal{H}}_s\\ =& \frac{1}{n-1}\sum_{i=1}^{n-1}B^{\mathcal{H}}_s(e_i,e_i),\end{aligned}$$

where $\{e_i\}_{i=1}^{n-1}$ is a local orthogonal frame for $\mathcal{H}$.

Corollary 2.8. *Thee mean curvature $\mu^{\mathcal{H}}$ has the following properties*

(a)

$$\begin{aligned}\mu^{\mathcal{H}} &= \frac{1}{n-1}(\mathrm{tr}S)N\\ &= -\frac{1}{2(n-1)}(\mathrm{tr}\mathcal{L}_N g)N.\end{aligned}$$

(b) $\mu^{\mathcal{H}}=0$, if and only if $\mathrm{tr}S=0$, also $\mathrm{tr}S=0$ if and only if $\mathrm{tr}\mathcal{L}_N g=0$.

The following definition is similar to the definition which was stated in [3]

Definition 2.9. *Let $\mathcal{H}$ be an $(n-1)$-dimensional distribution.*

(a) We call $\mathcal{H}$ totally umbilic if for all horizontal vector fields X, Y,

$$B^{\mathcal{H}}_s(X,Y)=\langle X,Y\rangle\,\mu^{\mathcal{H}}.$$

(b) We call $\mathcal{H}$ totally geodesic if for all horizontal vector fields X, Y,

$$B^{\mathcal{H}}_s(X,Y)=0.$$

(c) We call $\mathcal{H}$ minimal if

$$\mu^{\mathcal{H}}=0.$$

Based on the above definition, we conclude the following propositions.

Proposition 2.10. *The distribution $\mathcal{H}$ is totally geodesic, if and only if it is totally umbilic and minimal.*

Proposition 2.11. *The distribution $\mathcal{H}$ is totally umbilic, if and only if for each selection N, tensor S_s be a scalar. Especially $\mathcal{H}$ it is totally geodesic If and only if for each selection N, $S_s=0$.*

Proposition 2.12. *Let W be a nowhere zero Killing vector field. Then the distribution $\mathcal{H}=\langle W\rangle^{\perp}$ is totally geodesic and its symmetric shape operator is zero.*

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A discussion similar to the proposition 2.11 and knowing that the integrability of a distribution is equivalent to the symmetry of its second fundamental form (see [12] for more details), will lead to the following proposition.

Proposition 2.13. *Let $\mathcal{H}$ be an $(n-1)$-dimensional distribution.*

(a) The distribution $\mathcal{H}$ is integrable if and only if distribution shape operator, for each selection N, be symmetrical. Hence $S_s = S$ and $S_a = 0$.

(b) The distribution $\mathcal{H}$ is totally umbilic and integrable if and only if the distribution shape operator, for each choice N, be a scalar and therefore each leaf of its foliation is a totally umbilic hypersurface.

In the following, we assume that the vertical vector field N, be defined throughout manifold and therefore the shape operator obtained from N, without sign of ambiguously defined throughout manifold. Let $R^n(c)$ be a simply connected Riemannian space form with fixed sectional curvature c which is the Euclidean space $\mathbb{R}^n$ for $c = 0$, is the Euclidean sphere $\mathbb{S}^n$ for $c = 1$, and is the Hyperbolic space $\mathbb{H}^n$ for $c = -1$. A hypersurface in $R^n(c)$ is called an isoparametric if its principal curvatures are constant everywhere, counting multiplicities. In this case, we have the following proposition.

Proposition 2.14. *Let $\mathcal{H}$ be an $(n-1)$-dimensional distribution on $R^n(c)$ and $S = \alpha I$, where α is an arbitrary constant. Then either $c = 0$ or $c = -1$ and $\mathcal{H}$ is totally umbilic and integrable. If $c = 0$, then $\alpha = 0$, the distribution is totally geodesic, and each leaf of its foliation is a hyperplane. If $c = -1$, then $\alpha = \pm 1$, the distribution is non-minimal and each leaf of its foliation is a parabolic hypersurface.*

Corollary 2.15. *There is no $(n-1)$-dimensional distribution which is integrable and totally geodesic on $\mathbb{S}^n$ or $\mathbb{H}^n$.*

Proposition 2.16. *Let $\mathcal{H}$ be an $(n-1)$-dimensional distribution on $R^n(c)$, with $n > 2$, and $S = fI$, in which f is a smooth function on $R^n(c)$. Then either $c = 0$ or $c = -1$, and $\mathcal{H}$ is totally umbilic and integrable. If $c = 0$, then the distribution is totally geodesic and all leaves are hyperplanes and the function f is zero. If $c = -1$, then the distribution is non-minimal and each leaf is a hyperbolic or parabolic hypersurface. Also for all $x \in \mathbb{H}^n$, non-zero vector $a \in \mathbb{R}^{n+1}_1$, it is found that $\langle a, a\rangle \in \{0, 1\}$, and $f(x) = -\frac{\langle a,x\rangle}{\sqrt{\langle a,a\rangle + \langle a,x\rangle^2}}$.*

Corollary 2.17. *There is no $(n-1)$-dimensional distribution which is integrable and totally umbilic on $\mathbb{S}^n$, $n > 2$.*

Proposition 2.18. *There is no $(n-1)$-dimensional distribution which is integrable on $R^n(c)$ that its shape operator has only two distinct constant eigenvalues.*

Isoparameter hypersurfaces of $\mathbb{R}^n$ and $\mathbb{H}^n$ have at most two constant principal curvatures, and so, according to the propositions 2.14–2.18, we obtain the following result.

http://dx.doi.org/10.22108/msci.2023.138841.1600

Corollary 2.19. *The only $(n-1)$-dimensional distribution on $\mathbb{R}^n$, which is integrable and all eigenvalues of its shape operator are constant, is generated by $n-1$ linearly independent constant vector fields and so the distribution is totally geodesic and every leaf of its foliation is a hyperplane and the distribution shape operator is zero.*

Corollary 2.20. *There is no $(n-1)$-dimensional distribution on $\mathbb{H}^n$, which is integrable and all eigenvalues of its shape operator be constant and has at least two eigenvalues.*

3. Conclusion

In this paper, we studied the shape operator of a distribution and its properties, especially its decomposition into symmetric and antisymmetric parts. We showed that a distribution is totally umbilic if and only if the symmetric component of its shape operator is scalar, we also proved that it is totally geodesic if and only if the symmetric component of its form operator is zero. Finally, we considered the classification of $(n-1)$-dimensional distributions and studied simply connected space forms with a shape operator having at most two eigenvalues.

References

[1] L. J. Alías and S. M. B. Kashani, Hypersurfaces in space forms satisfying the condition $L_k x = Ax + b$, *Taiwanese J. Math.*, **14** no. 5 (2010) 1957–1977.

[2] I. Androulidakis and M. Zambon, Stefan-Sussmann singular foliations, singular subalgebroids and their associated sheaves, *Int. J. Geom. Methods Mod. Phys.*, **13** (2016) 17 pp.

[3] P. Baird and J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Mathematical Society Monographs. New Series, no. 29, The Clarendon Press, Oxford University Press, Oxford, 2003.

[4] F. Bullo and A. D. Lewis, *Geometric control of mechanical systems: modeling, analysis, and design for simple mechanical control systems*, **49**, Springer, 2019.

[5] T. E. Cecil and P. J. Ryan, Isoparametric hypersurfaces, in: Geometry of hypersurfaces, Springer, New York, (2015) 85–184.

[6] A. del Pino Gomez, Topological aspects in the study of tangent distributions,13th Young Researchers Workshop on Geometry, Mechanics and Control, 3–67, Textos Mat./Math. Texts, **48**, Univ. Coimbra, Coimbra, 2019.

[7] M. P. Do Carmo and J. Flaherty Francis, *Riemannian geometry*, **6**, Springer, 1992.

[8] O. Gil-Medrano, Geometric properties of some classes of Riemannian almost-product manifolds, *Rend. Circ. Mat. Palermo (2)*, **32** no. 3 (1983) 315–329.

[9] A. Gray, Pseudo-Riemannian almost product manifolds and submersions, *J. Math. Mech.*, **16** (1967) 715–737.

[10] N. J. Hicks, *Notes on differential geometry*, no. 3, D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto-London, 1965.

 http://dx.doi.org/10.22108/msci.2023.138841.1600

[11] J. M. Lee, *Introduction to Riemannian manifolds*, Second edition, Graduate Texts in Mathematics, **176**, Springer, Cham, 2018.

[12] M. C. Muñoz Lecanda, On some aspects of the geometry of non integrable distributions and applications, *J. Geom. Mech.*, **10** no. 4 (2018) 445–465.

[13] H. Nijmeijer and A. Van der Schaft, *Nonlinear dynamical control systems*, **464**, Springer-Verlag, New York, 1990.

[14] B. O'neill, *Semi-Riemannian geometry with applications to relativity*, Academic press, 1983.

[15] B. L. Reinhart, The second fundamental form of a plane field, *J. Differential Geometry*, **12** no. 4 (1977) 619–627.

[16] B. L. Reinhart, *Differential geometry of foliations. The fundamental integrability problem*, **99**, Springer-Verlag, Berlin, 1983.

[17] H. J. Sussmann, Orbits of families of vector fields and integrability of distributions, *Trans. Amer. Math. Soc.*, **180** (1973) 171–188.

Mehran Aminian

Department of Mathematics, Faculty of Mathematics, Vali-e-Asr University, Rafsanjan, Iran.

mehran.aminian@vru.ac.ir

Mehran Namjoo

Department of Mathematics, Faculty of Mathematics, Vali-e-Asr University, Rafsanjan, Iran.

namjoo@vru.ac.ir