

CONTINUOUS-ATTRACTOR IN RABINOVICH-FABRICANT SYSTEM AND ITS NOVEL GENERALIZED MODEL

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ABSTRACT. The aim of this work is the numerical study of the Rabinovich-Fabrikant system and its generalized model, which shows the occurrence of very rich dynamic behaviors with the interaction of three parameters of the generalized system. In particular, we observe the period-doubling bifurcation phenomenon leading to chaos, which has rarely been reported in previous works in the Rabinovich-Fabrikant system. The complex dynamic behaviors of the system are investigated by using the Lyapunov spectrum, the parameters dependent bifurcation diagram and different sections of the phase space. This study is based on the numerical solution of differential equations and their numerical bifurcation analysis using Matlab software. The obtained results are new, because the generalization of the Rabinovich-Fabrikant system of the current study was proposed and studied for the first time. The generalized model describes the three-mode interaction. It can be used to simulate systems in radio and electronics engineering in which there is a three-mode interaction and which include cubic nonlinear terms. In addition, although the Rabinovich-Fabrikant systems simulate systems of a physical nature, and in this regard, the coefficients embedded in them must be positive, its highly nonlinear and chaotic nature, due to the presence of third-order sentences, gives them a unique quality to be applied in secure communication. Resultantly, its artificial generalization with the use of negative parameters which adds to the complexity of its rich dynamics, is of particular importance.

Keywords: Rabinovich-Fabrikant system, generalized Rabinovich-Fabrikant system, chaotic attractors, Lyapunov exponents, bifurcation diagram.

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1. Introduction

A dynamic system is classified into two types, discrete and continuous, in terms of time evolution. In this paper, we will look at a particular continuous dynamical system known as the "Rabinovitch-Fabrikant system", which was invented in 1979 by two theoretical physicists, Mikhail Rabinovitch and Anatoly Fabrikant [1]. This system is actually a simplification of a complex nonlinear parabolic equation that models different physical systems such as Tollmien–Schlichting waves in hydrodynamic flows, wind waves on water, Langmuir waves in plasma. Having five equilibrium points, it is not topologically equivalent to many classical systems such as the Lorenz and Chen systems (with three equilibrium points), Rössler system (with two equilibrium points) and the like. In the following, we will call this system the RF system, for short.

There are several reasons why this system has been noticed. One is the fact that it models a physical system and, therefore, is not a synthetic model and can be used to model numerous physical phenomena. Another reason is that due to its strong nonlinearity (due to the presence of third-order terms in its mathematical model), a rigorous mathematical analysis cannot be performed on it, hence, the system may show new and interesting properties that have not yet been reported. At the same time, it creates serious challenges for the numerical methods of solving ordinary differential equations.

The mathematical model of the Rabinovich-Fabrikant (RF) system is described by the following equations:

$$(1.1) \quad \begin{aligned} \frac{dx_1}{dt} &= x_2 (x_3 - 1 + x_1^2) + ax_1 \\ \frac{dx_2}{dt} &= x_1 (3x_3 + 1 - x_1^2) + ax_2 \\ \frac{dx_3}{dt} &= -2x_3 (b + x_1x_2) \end{aligned}$$

In the Rabinovitch-Fabrikant (RF) system modeled by the system of ordinary differential equations (1.1), $a > 0$ and $b \in \mathbb{R}$ is the bifurcation parameter.

Since it is currently impossible to fully analyze this system mathematically, most of the research is based on numerical and computer analysis. Following this common practice, we will also, in this paper, have an approach based on numerical analysis.

2. Main Results

Different dedicated numerical methods for ODEs, implemented in different software packages, might give different results for the same parameters values and initial conditions. system (1.1) by Danca et al. [2, 3, 4] has been studied. By numerical analysis, they showed that this system exhibits unusual and very rich dynamics, including multistability. Here, we obtain their results with a different numerical

approach. They used the 3-step predictor–corrector Local Iterative Linearization (LIL) method, which is an implicit 3-step method [5].

In this paper, to calculate the Lyapunov exponents, we have used the MATDS software package applicable in Matlab software by adding codes to get the desired output, and to solve **ODE**, the *ode45* solver implemented in this software with relative error $RelTol = 10^{-7}$ and absolute error $AbsTol = 10^{-7}$ are used. The step length used in numerical methods is also considered to be $h = 10^{-6}$.

2.1. Generalized Rabinovitch-Fabricant system. Recently, a generalization of the system (1.1) has been presented and numerically investigated as follows [6, 8]:

$$\begin{aligned}\frac{dx_1}{dt} &= [p(x_1^2 + x_3) + q(-x_2^2 + 3x_3) - 1], x_2 + ax_1 \\ \frac{dx_2}{dt} &= [p(-x_1^2 + 3x_3) + q(x_2^2 + x_3) + 1], x_1 + ax_2 \\ \frac{dx_3}{dt} &= -2x_3(b + (p + q)x_1x_2).\end{aligned}$$

This system becomes the system (1.1) with $p = 1$ and $q = 0$. In this section, we introduce the following system as a generalization of the system (1.1):

$$\begin{aligned}(2.1) \quad \frac{dx_1}{dt} &= x_2(x_3 - 1 + x_1^2) + ax_1 \\ \frac{dx_2}{dt} &= x_1(3x_3 + 1 - x_1^2) + cx_2 \\ \frac{dx_3}{dt} &= -2x_3(b + x_1x_2)\end{aligned}$$

This system becomes system (1.1) with $c = a$. This generalization is simpler than generalization (2.1), but it includes the same results in the detection of chaos and especially the process of converting the bifurcation of periodic period to chaos. According to the relationship

$$\sum_{i=1}^3 \frac{\partial f_i}{\partial x_i} = (2x_1x_2 + a) + (c) + (-2x_1x_2 - 2b) = a + c - 2b,$$

The system (2.1) is dissipative whenever $a + c - 2b < 0$.

We fix the parameters $a = -1$ and $b = -0.1$ and consider c as the branching parameter. Therefore, with $c < 0.8$, the system will be dissipative and can be chaotic. Considering that system, (1.1) has a special application in encryption and secure communications, generalizing the system and adding to its complexity will achieve these goals and reduce the possibility of decryption and making communications unsafe. In the following, we examine the complex dynamics of the system for $a = -1$, $b = -0.1$ and some values of $c < 0.8$.

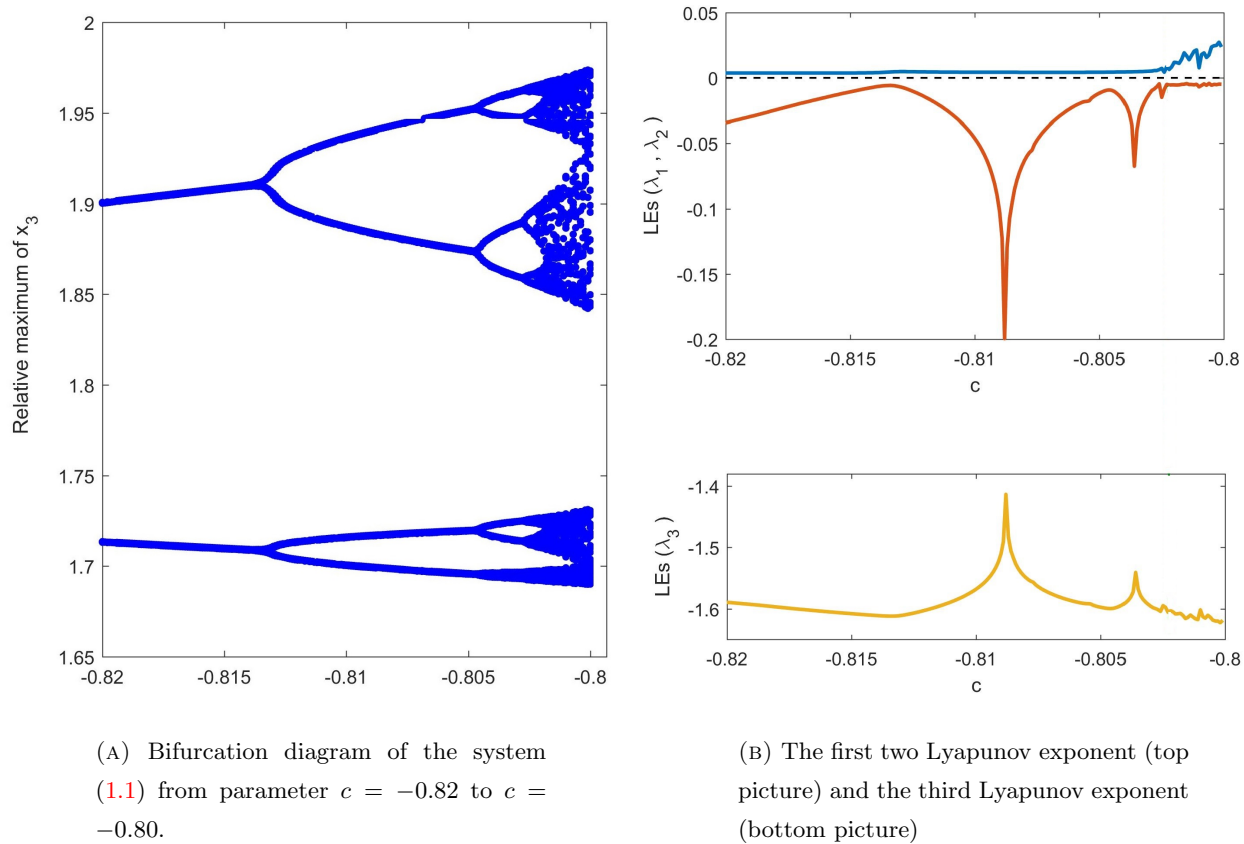
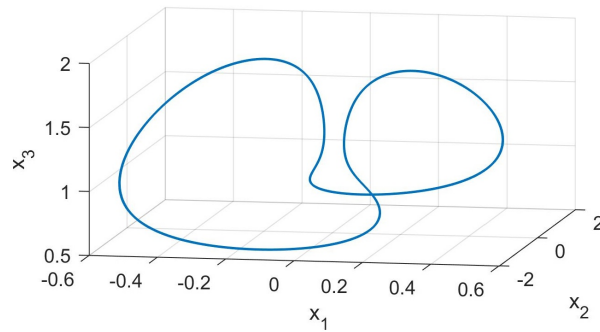


FIGURE 1. Bifurcation diagram and Lyapunov exponents of system (2.1).

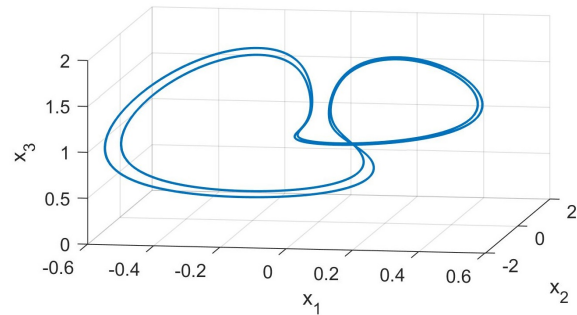
2.1.1. Period-doubling bifurcation route to chaos. By keeping $a = -1$ and $b = -0.1$ and changing c on the interval $[-0.82, -0.80]$, the bifurcation diagram and lyapunov exponents of the system (2.1), in the figure 1 are drawn. It can be seen that for $c \in [-0.8200, -0.8030]$ the system has two negative lyapunov exponents and one zero lyapunov exponent, and in this interval we have a Periodic Solutions, also for $c \in [-0.8030, -0.8000]$ We have a negative lyapunov exponent, a zero lyapunov exponent and a positive lyapunov exponent, so the system will be chaotic.

The bifurcation diagram drawn in figure 1a shows a period doubling route to chaos. In dynamical systems theory, a period-doubling bifurcation occurs when a slight change in a system's parameters causes a new periodic trajectory to emerge from an existing periodic trajectory. the new one having double the period of the original. This process is called period doubling, and its continuous repetition is one of the ways leading to chaos

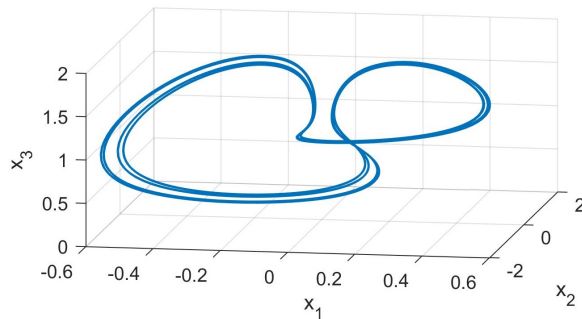
The bifurcation process of periodic period leading to chaos can be seen in figure 2.



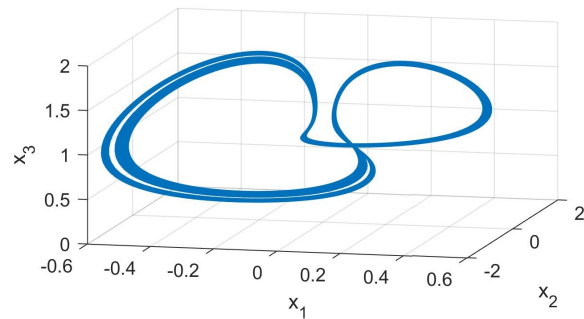
(A) $c = -0.82$. periodic solution with period 14.2.



(B) $c = -0.806$. periodic solution with period 28.4.



(C) $c = -0.8034$. periodic solution with period 56.7.



(D) $c = -0.8015$. Chaotic movement.

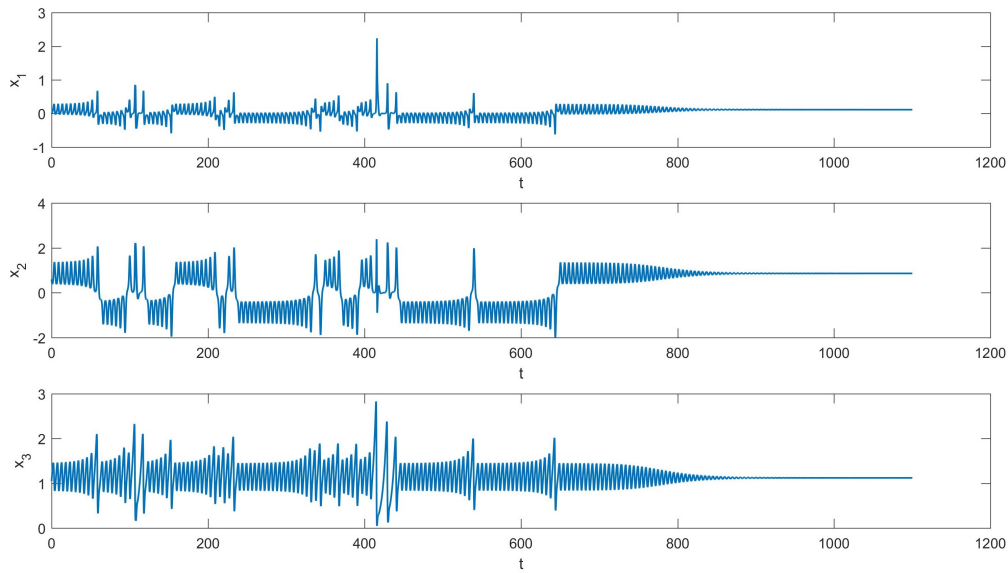
FIGURE 2. The phase space of the system (2.1) for $a = -1$, $b = -0.1$ and different values of c , which shows the process of converting period-doubling bifurcation to chaos.

2.1.2. transient chaos. A chaotic but transient behavior is observed in figure 3. As the diagram in figure 3a shows, this chaotic behavior lasts significantly, but after some time it disappears and leads to a fixed point.

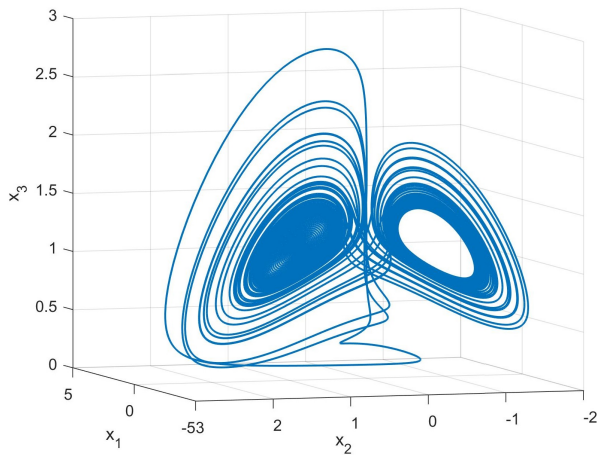
2.1.3. Coexistence of attractors. Figure 4 depicts the coexistence of some absorbers. Relevant details are specified in the figure captions.

3. Conclusions

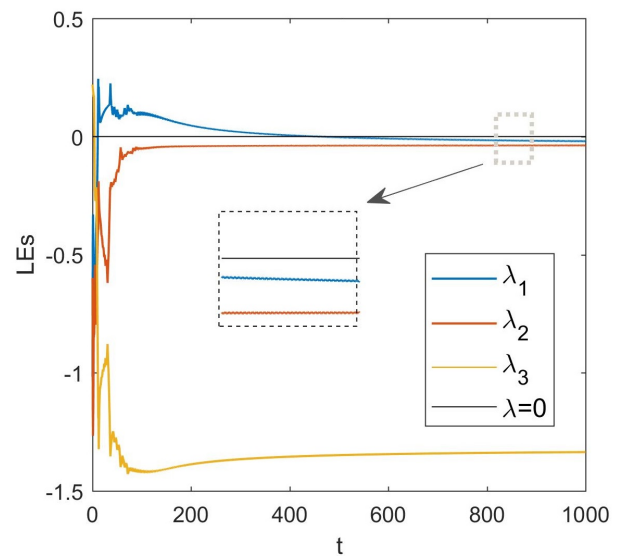
In this article, we studied the Rabinovich-Fabricant system and its generalized model numerically. For the generalized system, we determined the occurrence of very rich and complex dynamic behaviors with the interaction of three parameters. In particular, we showed the bifurcation phenomenon of



(A) The time response of state variables of the system (2.1).



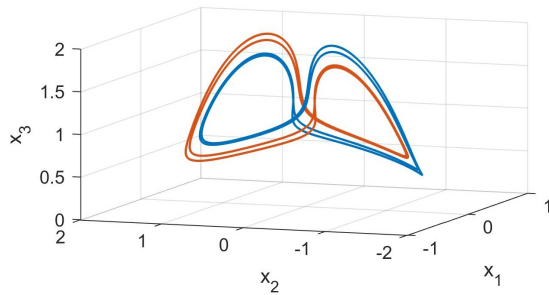
(B) The phase space of the system (2.1).



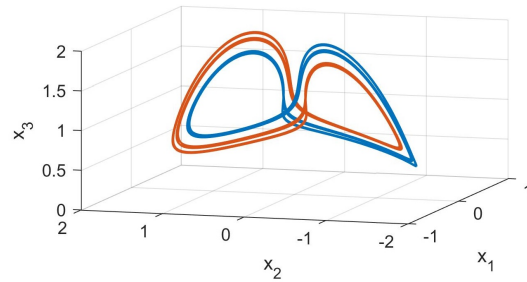
(C) Lyapunov expressions of the system (2.1).

FIGURE 3. Transient chaos in the (2.1) system for $a = -1$, $b = -0.1$ and $c = -0.5854$.

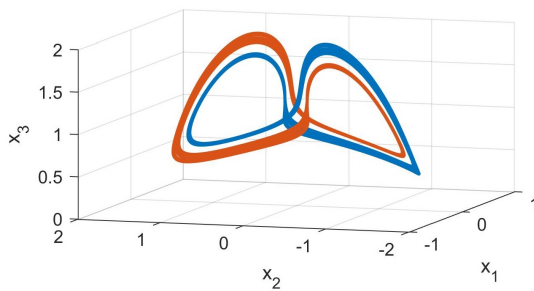
period doubling leading to chaos, which was not reported before in the Rabinovitch-Fabricant system. We have seen that by changing the bifurcation parameter in a very small interval, the evolution



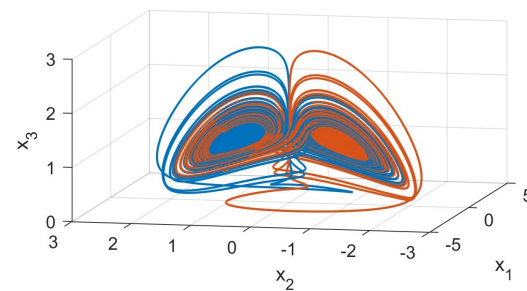
(A) $c = -0.806$. Coexistence of two periodic attractors with period 14.2.



(B) $c = -0.8034$. Coexistence of two periodic attractors with period 28.4.



(C) $c = -0.8011$. Coexistence of two chaotic attractors.



(D) $c = -0.5854$. Coexistence of two transient chaotic attractors.

FIGURE 4. Coexistence of attractors for $a = -1$, $b = -0.1$ and some values of c .

process of the system, with the continuous repetition of bifurcation bifurcation of periodic period, ends from periodic solution to chaos. The complex dynamic behaviors of the system were investigated using the Lyapunov exponents, the bifurcation diagram depending on the parameters and different sections of the phase space. This study was based on the numerical solution of differential equations and their numerical bifurcation analysis using Matlab software. The obtained results are new, as this generalization of the Rabinovitch-Fabrikant system is proposed for the first time. The generalized model can be used to simulate systems in radio and electronics engineering where there is a three-mode interaction and which include third-order nonlinear terms. The Rabinovitch-Fabricant system simulate systems of a physical nature, and in this regard, the coefficients embedded in it must be positive, but its highly nonlinear and chaotic nature, arising from the presence of third-order sentences, gives them a unique quality to be applied in secure communication. Therefore, its artificial generalization, using negative parameters, which itself adds to the complexity of its rich dynamics, is of particular importance.

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