

A SUBCLASS OF STRONGLY STARLIKE FUNCTIONS

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ABSTRACT. Let's denote $\mathcal{S}^*(f_c)$ as a family of analytic functions $f(z) = z + a_2z^2 + a_3z^3 + \dots$ in the open unit disk $\mathbb{D}$ that satisfy the following relation for $c \in (0, 1)$:

$$\frac{zf'(z)}{f(z)} \prec f_c(z) = \frac{1}{\sqrt{1-cz}}, \quad z \in \mathbb{D}.$$

First, we introduce the analytic functions $f_c(z)$ and examine their starlike and positivity properties of the real part. Then, we obtain their images in the open unit disk $\mathbb{D}$, which are Cassini ovals. Cassini ovals, due to their properties, have applications in solving various problems in fields such as geometry, physics, and mathematics. These curves are used in studying the motion of waves and electromagnetic waves in interstellar spaces, as well as in the design of engineering structures such as telescopes. In this article, with the help of integrals, we investigate the structure of mappings in this family and some properties including maximum and minimum moduli, bounds of the real part of these functions. Moreover, we obtain the relationships between the defined geometric ranks with this family, including the order of starlikeness and order of strong starlikeness.

1. Introduction

Let $\mathcal{A}$ be a set of analytic functions of the form $f(z) = z + a_2z^2 + a_3z^3 + \dots$ in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathcal{A}$ is called univalent if it is one-to-one.

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In [5], two classes of starlike and convex functions with order $0 \leq \beta < 1$ are defined as follows:

$$(1.1) \quad \mathcal{S}^*(\beta) := \left\{ f \in \mathcal{A}: \Re \left(\frac{zf'(z)}{f(z)} \right) > \beta \right\}, \quad \mathcal{K}(\beta) := \{ f \in \mathcal{A}: zf'(z) \in \mathcal{S}^*(\beta) \}.$$

Similarly, in [2], the class of functions called strongly starlike with order $0 < \alpha \leq 1$ is defined as:

$$\mathcal{SS}^*(\alpha) = \left\{ f \in \mathcal{A}: \left| \operatorname{Arg} \frac{zf'(z)}{f(z)} \right| < \frac{\alpha\pi}{2} \right\}.$$

If f and g are two analytic functions in $\mathbb{D}$, we say that f is subordinate to g [3], denoted by $f \prec g$, if and only if there exists an analytic function w with $w(0) = 0$ such that for all $z \in \mathbb{D}$:

$$|w(z)| < 1, \quad f(z) = g(w(z)).$$

If g is univalent, we have:

$$f(z) \prec g(z) \iff f(0) = g(0), \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

Given $c \in (0, 1)$, analytic functions f_c are defined as follows:

$$(1.2) \quad f_c(z) := \frac{1}{\sqrt{1-cz}} = 1 + \frac{c}{2}z + \frac{3c^2}{8}z^2 + \dots$$

in the principal branch of the complex logarithm, where $\log 1 = 0$. These functions are univalent in $\mathbb{D}$ and map the open unit disc $\mathbb{D}$ into the interior of the Cassinian ovals given by the Cartesian equation:

$$(1.3) \quad (x^2 + y^2)^2 - \frac{2}{1-c^2}(x^2 - y^2) + \frac{1}{1-c^2} = 0,$$

or the polar equation:

$$(1.4) \quad r^4 - \frac{2r^2}{1-c^2} \cos(2\theta) = \frac{1}{c^2-1}.$$

In Figure 1, the boundary of $f_c(\mathbb{D})$ and some of its properties are shown.

2. Main Results

In this section, we will first derive the structure of functions in the class $\mathcal{S}^*(f_c)$, and then using the stated theorems, we will determine the order of starlikeness and strongly starlikeness of functions in the class $\mathcal{S}^*(f_c)$.

Theorem 2.1. *A function f belongs to the class $\mathcal{S}^*(f_c)$ if and only if there exists a function $p \prec f_c$ such that*

$$(2.1) \quad f(z) = z \exp \left(\int_0^z \frac{p(t)-1}{t} dt \right), \quad z \in \mathbb{D}.$$

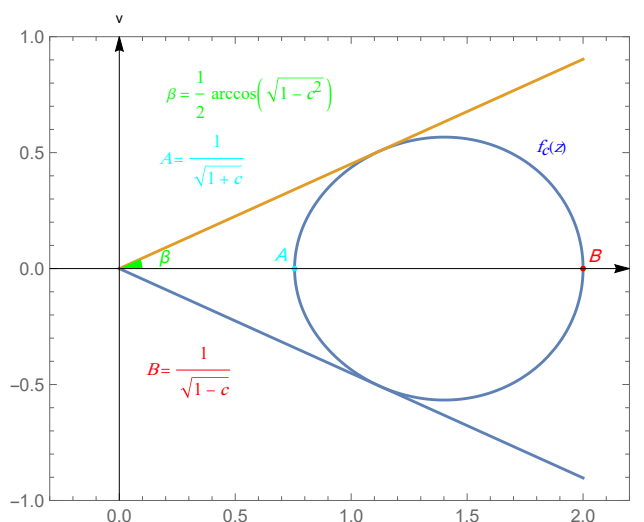


FIGURE 1. The boundary of $f_c(\mathbb{D})$

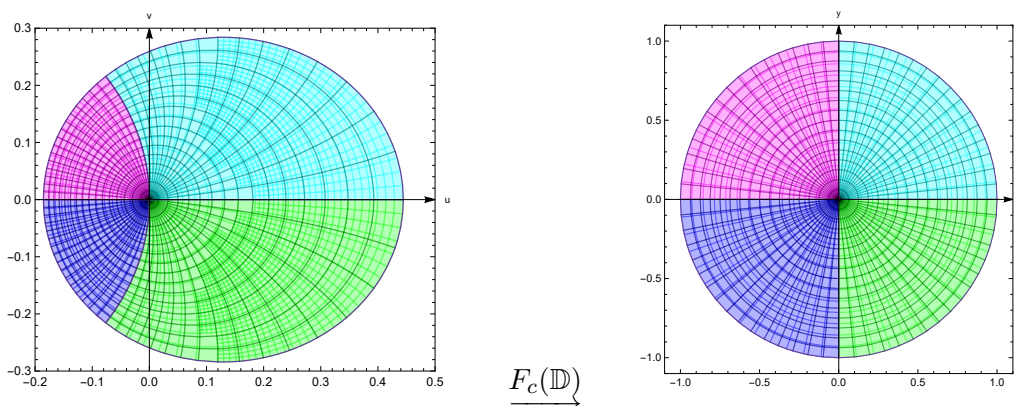


FIGURE 2. The image of the open unit disk $\mathbb{D}$ under the mapping $F_c(z)$.

If we set $p(z) = f_c(z)$ in theorem (2.1), then we get

$$(2.2) \quad F_c(z) := z \exp \left(\int_0^z \frac{f_c(t) - 1}{t} dt \right) = \frac{4z}{(1 + \sqrt{1 - cz})^2}, \quad z \in \mathbb{D}.$$

This function $F_c(z)$ is an extreme function for the class $\mathcal{S}^*(f_c)$. Figure 2 illustrates the image of the open unit disk $\mathbb{D}$ under the mapping $F_c(z)$ for $c = 3/4$.

Theorem 2.2. Let f_c be the given function described in (1.2). Then f_c is convex and satisfies the following conditions:

$$(2.3) \quad \max_{|z|=r<1} |f_c(z)| = f_c(r), \quad \min_{|z|=r<1} |f_c(z)| = f_c(-r).$$

In the following theorem, we obtain bounds for the real part and strongly starlike mappings of the functions f_c .

Theorem 2.3. Suppose $c \in (0, 1)$. Then we have the following:

(1)

$$f_c(\mathbb{D}) \subset \left\{ w \in \mathbb{C} : \frac{1}{\sqrt{1+c}} < \Re(w) < \frac{1}{\sqrt{1-c}} \right\},$$

(2)

$$f_c(\mathbb{D}) \subset \left\{ w \in \mathbb{C} : |\text{Arg}(w)| < \frac{1}{2} \arccos \sqrt{1-c^2} \right\}.$$

Theorem 2.4. If $f \in \mathcal{S}^*(f_c)$ and $|z| = r < 1$, then the following hold:

(1)

$$\frac{zf'(z)}{f(z)} \prec \frac{zF'_c(z)}{F_c(z)}, \quad \frac{f(z)}{z} \prec \frac{F_c(z)}{z},$$

(2)

$$F'_c(-r) \leq |f'(z)| \leq F'_c(r),$$

(3)

$$-F_c(-r) \leq |f(z)| \leq F_c(r),$$

(4)

$$|\arg(f(z)/z)| \leq \max |z| = r \arg \left(\frac{1}{(1 + \sqrt{1-cz})^2} \right),$$

(5) Either f is a rotation of F_c or

$$\left\{ w \in \mathbb{C} : |w| \leq -F_c(-1) = \frac{4}{(1 + \sqrt{1+c})^2} \right\} \subset f(\mathbb{D}),$$

where in all cases, the function F_c is defined as per equation (2.2).

In the following theorem, we determine the subordination order and strong subordination order for the class of functions $\mathcal{S}^*(f_c)$.

Theorem 2.5. The class of functions $\mathcal{S}^*(f_c)$ has the following properties:

(1) For $0 \leq \beta \leq \frac{1}{\sqrt{1+c}}$, we have

$$\mathcal{S}^*(f_c) \subset \mathcal{S}^*(\beta).$$

(2) For $\frac{1}{\pi} \arccos \sqrt{1-c^2} \leq \alpha \leq 1$, we have

$$\mathcal{S}^*(f_c) \subset \mathcal{SS}^*(\alpha).$$

3. Conclusions

The class $\mathcal{S}^*(f_c)$ consists of functions that can be represented in a specific form involving the function f_c , which is a special function related to the starlikeness property. The function $F_c(z)$, derived from $f_c(z)$, is an extreme function for the class $\mathcal{S}^*(f_c)$ and has specific properties, including convexity and bounds on its maximum and minimum modulus on the unit circle.

The presented theorems provide bounds for the real part of the functions f_c and establish relationships related to subordination and strong subordination order for the class of functions $\mathcal{S}^*(f_c)$. Overall, the obtained theorems and their proofs contribute to understanding the structural properties, order of starlikeness and strongly starlikeness, as well as subordination order within the class of functions $\mathcal{S}^*(f_c)$, for different values of the parameter c .

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