http://math-sci.ui.ac.ir

Journal of Mathematics and Society
ISSN (print): 2345-6493, ISSN (on-line): 2345-6507
Vol. 8 No. 4 (2023), pp. 71-79.

http://www.ui.ac.ir

THE CLIQUE NUMBER OF THE INTERSECTION GRAPH OF A CYCLIC GROUP OF ORDER WITH AT MOST THREE PRIME FACTORS

SEYYED MAJID JAFARIAN AMIRI* AND AREZOO BEHESHTIPOUR

Abstract. Let G be a finite non-trivial group. The intersection graph $\Gamma(G)$, is a graph whose vertices are all proper non-trivial subgroups of G, and there is an edge between two distinct vertices H and K if and only if $H \cap K \neq 1$. In this paper, we determine the clique number of the intersection graph of the cyclic groups of orders having at most three primes in their decomposition.

1. Introduction

Let G be a finite group. There are several ways to associate a graph to G (see [7] and the references therein). The one we will consider in this paper, is denoted by $\Gamma(G)$ and is called the intersection graph of G. The intersection graph $\Gamma(G)$ of a nontrivial group G is an undirected graph without loops and multiple edges defined as follows: the vertex set is the set of all proper non-trivial subgroups of G, and there is an edge between two distinct vertices H and K if and only if $H \cap K \neq 1$ where 1 denotes the trivial subgroup of G. The graph $\Gamma(G)$ has been extensively studied (see, for example, [1, 8, 11, 12]).

Currently, the present authors in [4], have determined all groups G such that the clique number of $\Gamma(G)$ is less than 5, and also they have given a criterion for solvability of finite groups G, by the clique

Keywords: Finite group, Cyclic group, Intersection graph, Clique number.

Communicated by Saeid Maghsoudi.

Article Type: Resaerch Paper.

*Corresponding author.

Received: 24-10-2023, Accepted: 20-11-2023, Published Online: 31-12-2023.

Cite this article: S. M. Jafarian Amiri and A. Beheshtipour, The clique number of the intersection graph of a cyclic group of order with at most three prime factors, *Journal of Mathematics and Society*, **8** no. 4 (2023) 71–79.

http://dx.doi.org/10.22108/msci.2023.139575.1618 .

number of $\Gamma(G)$. More precisely, it has been proved that if G is a finite group such that the clique number of $\Gamma(G)$ is less than 13, then G is solvable. Note that 13 is the clique number of $\Gamma(A_5)$, where A_5 is the alternating group on 5 letters.

Other researches in this topic are intersection graphs of a semigroup, a module, and ideals of a ring, were investigated in [5], [2] and [6, 10], respectively.

2. Main Results

We start this section with the following definition:

Definition 2.1. *The subset X of vertices of a finite graph Γ is called a clique, if the induced subgraph by X is a complete graph. The maximum size of a clique, among all cliques of Γ, is called the clique number of Γ and we denote it by $\omega(\Gamma)$. If Γ is empty (without vertex), then we define $\omega(\Gamma) = 0$ and $\omega(\Gamma) = 1$ if Γ is null (with a non-empty vertex set with no edges). A clique X in Γ is called maximal if there is no clique Y in Γ such that $X \subsetneq Y$.*

Note that the maximum size, among all maximal cliques in $\Gamma(G)$, is $\omega(\Gamma(G))$.

To prove the main theorems, we need the following result that its proof can be found in the most valid book of group theory.

Proposition 2.2. *If $G = \langle g \rangle$ is a cyclic group of ordsr n, then for any divisor s of n, there is a unique subgroup $H = \langle g^{\frac{n}{s}} \rangle$ of G of order s.*

The following result is a consequence of the above proposition.

Corollary 2.3. *Let G be a finite cyclic group. Then, the intersection of two proper subgroups of G is non-trivial if and only if their orders are not relatively prime.*

For a natural number n, we denote by C_n the cyclic group of order n, $\pi(n)$ the set of prime divisors of n and $d(n)$ the number of all divisors of n. Note that if p is a prime number and n is a multiple of p, then the number of divisors of n with multiple p is $d(\frac{n}{p})$. If $V(\Gamma(G))$ is the set of vertices of $\Gamma(G)$, then by Proposition 2.2, we have $|V(\Gamma(C_n))| = d(n) - 2$. In this paper, we obtain $\omega(\Gamma(C_n))$, where $|\pi(n)| = 3$.

3. Summary of Proofs/Conclusions

Now, we state and prove our main results. First, we find the clique number of a cyclic group of a prime power order.

Proposition 3.1. *If p is a prime and m is a positive integer, then $\omega(\Gamma(C_{p^m})) = m - 1$.*

Proof. Since $|V(\Gamma(C_n))| = d(n) - 2$ and $d(p^m) = m + 1$, we get the conclusion. □

http://dx.doi.org/10.22108/msci.2023.139575.1618

In the sequel, assume that p_1, p_2 and p_3 are distinct primes. Also assume that n_1, n_2 and n_3 are positive integers such that $n_1 \geq n_2 \geq n_3$.

In the following results, we obtain the clique number of the intersection graph of group $C_{p_1^{n_1} p_2^{n_2}}$. We recall that $d(p_1^{n_1} p_2^{n_2}) = (n_1+1)(n_2+1)$.

Proposition 3.2. *We have* $\omega(\Gamma(C_{p_1^{n_1} p_2^{n_2}})) = d(p_1^{n_1} p_2^{n_2}) - 2 - d(p_2^{n_2}) = n_1 n_2 + n_1 - 1$.

Proof. Suppose that $G = C_{p_1^{n_1} p_2^{n_2}}$. Then, we define the subsets of $V(\Gamma(G))$ as follows:

- For $1 \leq i \leq 2$, $V_{p_i}(\Gamma(G))$ is the set of all proper subgroups H of G such that $\pi(|H|) = \{p_i\}$.
- $V_{p_1 p_2}(\Gamma(G))$ is the set of all proper subgroups H of G such that $\pi(|H|) = \{p_1, p_2\}$.

It is clear that $\{V_{p_1}(\Gamma(G)), V_{p_2}(\Gamma(G)), V_{p_1 p_2}(\Gamma(G))\}$ forms a partition for $V(\Gamma(G))$.

By Proposition 2.2, we have $|V_{p_i}(\Gamma(G))| = d(p_i^{n_i}) - 1 = n_i$ and $|V_{p_1 p_2}(\Gamma(G))| = d(\frac{n}{p_1 p_2}) - 1 = n_1 n_2 - 1$. By Corollary 2.3, in $\Gamma(G)$, each element of the clique $V_{p_1}(\Gamma(G))$ does not join to any element of the clique $V_{p_2}(G)$ and moreover all elements of $V_{p_i}(\Gamma(G))$ for $i = 1, 2$ join to all elements of the clique $V_{p_1 p_2}(G)$. Therefore $V_{p_1}(\Gamma(G)) \cup V_{p_1 p_2}(\Gamma(G))$ and $V_{p_2}(\Gamma(G)) \cup V_{p_1 p_2}(\Gamma(G))$ are the only maximal cliques of $\Gamma(G)$ and since $n_1 \geq n_2$, we have the result. □

Now we state the last main result.

Theorem 3.3. *Let* $G = C_n$ *where* $n = p_1^{n_1} p_2^{n_2} p_3^{n_3}$. *Then*

(1) *If* $n_1 \geq n_2 n_3$, *then* $\omega(\Gamma(G)) = d(\frac{n}{p_1}) - 1 = n_1 + n_1 n_2 + n_1 n_3 + n_1 n_2 n_3 - 1$.

(2) *If* $n_1 \leq n_2 n_3$, *then* $\omega(\Gamma(G)) = d(\frac{p_1^{n_1} p_2^{n_2}}{p_1 p_2}) + d(\frac{p_1^{n_1} p_3^{n_3}}{p_1 p_3}) + d(\frac{p_2^{n_2} p_3^{n_3}}{p_2 p_3}) + d(\frac{n}{p_1 p_2 p_3}) - 1$

$$= n_1 n_2 + n_1 n_3 + n_2 n_3 + n_1 n_2 n_3 - 1.$$

Proof. Similar to the proof of Proposition 3.2, we define the subsets of $V(\Gamma(G))$ as follows:

- For $1 \leq i \leq 3$, $V_{p_i}(\Gamma(G))$ is the set of all subgroups H of G such that $\pi(|H|) = \{p_i\}$. Therefore, $|V_{p_i}(\Gamma(G))| = d(p_i^{n_i}) - 1 = n_i$.
- $V_{p_i p_j}(\Gamma(G))$ is the set of all subgroups H of G such that $\pi(|H|) = \{p_i, p_j\}$ for $1 \leq i < j \leq 3$. Therefore, $|V_{p_i p_j}(\Gamma(G))| = d(\frac{p_i^{n_i} p_j^{n_j}}{p_i p_j}) = n_i n_j$ where $i \neq j$.
- $V_{p_1 p_2 p_3}(\Gamma(G))$ is the set of all proper subgroups H of G such that $\pi(|H|) = \{p_1, p_2, p_3\}$. So, we have $|V_{p_1 p_2 p_3}(\Gamma(G))| = d(\frac{n}{p_1 p_2 p_3}) - 1 = n_1 n_2 n_3 - 1$.

By Proposition 2.2, the above sets forms a partition for $V(\Gamma(G))$. By Corollary 2.3, in $\Gamma(G)$, each element of the clique $V_{p_i}(\Gamma(G))$ does not join to any element of the clique $V_{p_j}(G) \cup V_{p_j p_k}(G)$ for all distinct i, j, k and moreover all elements of $V_{p_i}(\Gamma(G))$ for $i = 1, 2, 3$, join to all elements of the clique $V_{p_1 p_2 p_3}(G) \cup V_{p_i p_j}(G)$, where $1 \leq i \neq j \leq 3$. Since $|G|$ has three prime divisors, the intersection of every two subgroups of G of orders with two distinct prime divisors, is non-trivial (by Corollary 2.3).

http://dx.doi.org/10.22108/msci.2023.139575.1618

It follows that $V_{p_1p_2}(\Gamma(G)) \cup V_{p_1p_3}(\Gamma(G)) \cup V_{p_2p_3}(\Gamma(G))$ is a cloque in $\Gamma(G)$. Now, we define W_i as follows for $1 \leq i \leq 4$:

$$W_1 = V_{p_1}(\Gamma(G)) \cup V_{p_1p_2}(\Gamma(G)) \cup V_{p_1p_3}(\Gamma(G)) \cup V_{p_1p_2p_3}(\Gamma(G)), \quad |W_1| = n_1 + n_1n_2 + n_1n_3 + n_1n_2n_3 - 1$$

$$W_2 = V_{p_2}(\Gamma(G)) \cup V_{p_1p_2}(\Gamma(G)) \cup V_{p_2p_3}(\Gamma(G)) \cup V_{p_1p_2p_3}(\Gamma(G)), \quad |W_2| = n_2 + n_1n_2 + n_2n_3 + n_1n_2n_3 - 1$$

$$W_3 = V_{p_3}(\Gamma(G)) \cup V_{p_1p_3}(\Gamma(G)) \cup V_{p_2p_3}(\Gamma(G)) \cup V_{p_1p_2p_3}(\Gamma(G)), \quad |W_3| = n_3 + n_1n_3 + n_2n_3 + n_1n_2n_3 - 1$$

$$W_4 = V_{p_1p_2}(\Gamma(G)) \cup V_{p_1p_3}(\Gamma(G)) \cup V_{p_2p_3}(\Gamma(G)) \cup V_{p_1p_2p_3}(\Gamma(G)), \quad |W_4| = n_1n_2 + n_1n_3 + n_2n_3 + n_1n_2n_3 - 1.$$

It is easy to see that W_1, W_2, W_3 and W_4 are the only maximal cliques in $\Gamma(G)$. Therefore,

$$\omega(\Gamma(G)) = max\{|W_1|, |W_2|, |W_3|, |W_4|\}.$$

Since $n_1 \geq n_2 \geq n_3$, we have $|W_1| \geq |W_2| \geq |W_3|$. Thus

$$\omega(\Gamma(G)) = max\{|W_1|, |W_4|\} = max\{n_1 + n_1n_2 + n_1n_3 + n_1n_2n_3 - 1, n_1n_2 + n_1n_3 + n_2n_3 + n_1n_2n_3 - 1\},$$

this completes the proof. □

References

[1] H. Ahmadi and B. Taeri, Planarity of the intersection graph of subgroups of a finite groups, *J. Algebra Appl.*, **15** no. 3 (2016) 19 pp.

[2] S. Akbari, F. Heydari and M. Maghasedi, The intersection graph of a group, *J. Algebra Appl.*, **14** no. 5 (2015) 9 pp.

[3] S. Akbari, H. A. Tavallayee and S. Khalashi Ghezelahmad, Intersection graph of submodules of a module, *J. Algebra Appl.*, **11** no. 1 (2012) 8 pp.

[4] A. Beheshtipour and S. M. Jafarian Amiri, The clique number of the intersection graph of a finite group, *Bull. Iranian Math. Soc.*, **49** no. 5 (2023) 16 pp.

[5] J. Bosák, *Theory of Graphs and its Applications*, (Proc. Sympos. Smolenice, 1963), Publ. House Czech. Acad. Sci., Prague, 1964 119–125.

[6] I. Chakrabarty, S. Ghosh, T. K. Mukherjee and M. K. Sen, Intersection graphs of ideals of rings, *Discrete Math.*, **309** no. 17 (2009) 5381—5392.

[7] P. J. Cameron, Graphs defined on groups, *Int. J. Group Theory*, **11** no. 2 (2022) 53–107.

http://dx.doi.org/10.22108/msci.2023.139575.1618

[8] B. Csákány and G. Pollák, The graph of subgroups of a finite group, (Russian) *Czechoslovak Math. J.*, **19** (1969) 241—247.

[9] X. Ma, On the diameter of the intersection graph of a finite simple group, *Czechoslovak Math. J.*, **66** no. 2 (2016) 365–370.

[10] F. Moh'd, M. Ahmed, Simple-intersection graphs of rings, *AIMS Math.*, **8** no. 1 (2023) 1040-1054.

[11] R. Shen, Intersection graphs of subgroups of finite groups, *Czechoslovak Math. J.*, **60** no. 4 (2010) 945–950.

[12] B. Zelinka, Intersection graphs of finite abelian groups, *Czechoslovak Math. J.*, **25** (1975) 171—174.

[13] A. Beheshtipour and S. M. Jafarian Amiri, The clique number of the intersection graph of abelian groups of order pqr, *54th Conferences of Iranian Mathematical Society (Zanjan)*, (1402) 182–185. [In persian]

Seyyed Majid Jafarian Amiri

Department of Mathematics, University of Zanjan, P. O. Box 45371-38791, Zanjan, Iran

Email: sm_jafarian@znu.ac.ir

Arezoo Beheshtipour

Department of Mathematics, University of Zanjan, P. O. Box 45371-38791, Zanjan, Iran

Email: arezoo.b1010@yahoo.com