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# ON THE MELNIKOV FUNCTION

MAJID KARIMI AMALEH

ABSTRACT. In this article, we introduce one of the most important topics in the subject of dynamical systems, namely the Melnikov function, in simple language. Melnikov function is one of the tools that can express the effect of the small perturbation on the homoclinic orbits. This issue is also related to breaking a homoclinic orbit under the effect of some small perturbations. When a small perturbation occurs in a dynamical system, some dynamical behaviors of the system may change. Here, we try to explain how to compute the formula of this function fluently. Therefore, in the first part, we will introduce some preliminary concepts and properties, and then in the second part, we will describe the construction of the Melnikov function. Finally, by stating the results of the fundamental matrix solutions and then using them, we will construct the Melnikov function near a homoclinic or hetroclinic orbit.

## 1. Introduction

One of the fundamental issues that have recently attracted the attention of researchers in the field of theoretical research is the issue of dynamical systems. The phenomenon of touching stable and unstable manifolds is an example of behavior in a dynamical system that results in the existence of special solutions. If the stable and unstable curves of an equilibrium point touch each other, then the solution is homoclinic, and if the stable curves of one equilibrium point intersect with the unstable curve of another equilibrium point, then we will have a heteroclinic solution. Investigating the existence of such solutions for a specific dynamical system has always been one of the important issues, and

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therefore many people have done good research in this field [8, 4, 3]. In addition, many researchers have tried to solve certain cases of Hilbert's 16th problem using the Melnikov function [2, 5, 6]. The subject of this article is related to the breaking of a homoclinic orbit. When a perturbation occurs in a dynamic system, some dynamical behaviors of the system may change, say, homoclinic orbit may break in the phase space. One of the tools that, can express the effect of small perturbations on the homoclinic orbit, is the Melnikov function, which in this article we try to explain how to make it fluently.

In the next section, we first give some basic definitions, and introduce some preliminary concepts, and then, we will describe the construction of the Melnikov function.

## 2. Main Results

We consider the differential equation $\dot{x} = f(x, \mu)$, where $x \in \mathbb{R}^2$, $\mu \in \mathbb{R}$ and $f$ is $C^2$. Suppose there are two hyperbolic saddles $p_-(\mu)$ and $p_+(\mu)$ and that for $\mu = 0$ there is a solution of this equation in the form of $x_*(t)$ such that

$$\lim_{t\to-\infty} x_*(t) = p_-(0), \ \lim_{t\to+\infty} x_*(t) = p_+(0).$$

Our goal is to check that when $\mu$ changes, how this connection (that is, the saddle connection between $p_-(t)$ and $p_+(t)$) is broken.

We put $x_*(0) = x_0$. The velocity vector $x_*(t)$ at $t = 0$ is:

$$\dot{x}_*(0) = f(x_0, 0) = (f_1(x_0, 0), f_2(x_0, 0)).$$

If we put

$$u_0 = \frac{1}{\|f(x_0, 0)\|^2}(-f_2(x_0, 0), f_1(x_0, 0)),$$

then obviously, the vector $u_0$ is perpendicular to $f(x_0, 0)$. We consider a line segment $\Sigma$, passing through $x_0$ and in the direction of $u_0$. Therefore, $\Sigma$ with the formula $x = x_0 + \xi u_0$ where $|\xi| < \alpha$ for some $\alpha > 0$. Suppose $x_-(t, \mu)$ is a solution of $\dot{x} = f(x, \mu)$ such that

$$(1)\ x_-(0, \mu) \in \Sigma,$$

$$(2)\ \lim_{t\to-\infty} x_-(t, \mu) = p_-(\mu),$$

$$(3)\ x_-(t, 0) = x_*(t).$$

This answer is on the unstable manifold $p_-(\mu)$. Similarly, suppose that $x_+(t, \mu)$ is a solution of $\dot{x} = f(x, \mu)$ such that


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$$(1)\ x_+(0,\mu) \in \Sigma,$$

$$(2)\ \lim_{t\to+\infty} x_+(t,\mu) = p_+(\mu),$$

$$(3)\ x_+(t,0) = x_*(t).$$

This solution is also placed inside the stable manifold $p_+(\mu)$ (pay attention to the Figures 1 and 2).

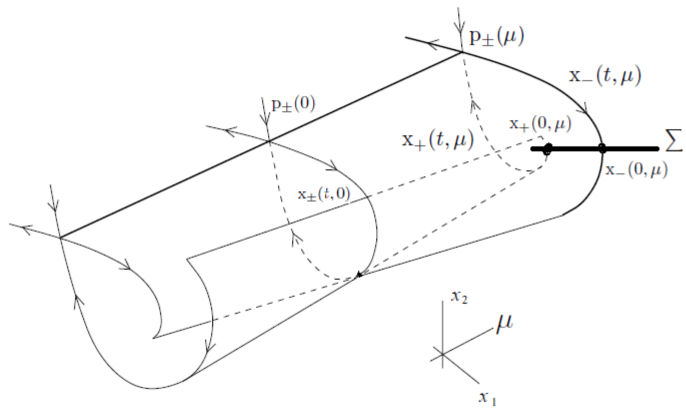


FIGURE 1. The conditions of the problem for the case where two hyperbolic saddles coincide, that is, $p_+ = p_-$

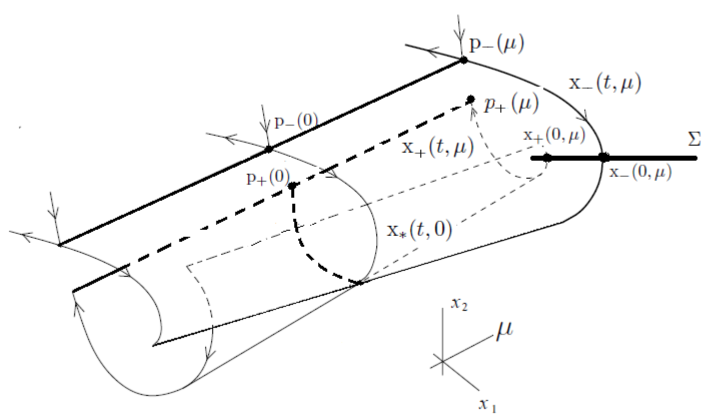


FIGURE 2. The conditions of the problem for the case where two hyperbolic saddles do not coincide, that is, $p_+ \neq p_-$

We define the separation function as follows:

$$s(\mu) = \xi_-(\mu) - \xi_+(\mu).$$

We put $\psi_0 = (-f_2(x_0, 0), f_1(x_0, 0))$, then $\dot{x}_*(t) = (\dot{x}_{*1}(t), \dot{x}_{*2}(t))$ is a solution of the equation $\dot{v}(t) = A(t)v$ and so,

$$\psi(t) = exp\left(-\int_0^t div f(x_*(s), 0)ds\right)(-\dot{x}_{*2}(t), \dot{x}_{*1}(t)),$$

is a solution of the adjoint equation $\dot{w} = -wA(t)$, with the initial condition

$$\psi(0) = (-\dot{x}_{*2}(0), \dot{x}_{*1}(0)) = (-f_2(x_0, 0), f_1(x_0, 0)) = \psi_0.$$

Since $\psi_0 u_0 = 1$, it follows that,

$$\frac{d\xi_\pm}{d\mu}(0) = \psi_0 \frac{d\xi_\pm}{d\mu}(0)u_0 = \psi_0 \frac{\partial x_\pm}{\partial \mu}(0, 0).$$

Also $\frac{\partial x_\pm}{\partial \mu}(t, 0)$ is a solution of the following equation:

$$\dot{v} = D_x f(x_*(t), 0)v + \frac{\partial f}{\partial \mu}(x_*(t), 0). \tag{2.1}$$

Assume that the state transition matrix of the homogeneous linear system

$$\dot{v} = D_x f(x_*(t), 0)v,$$

is $\phi(t, s)$. Now, according to the parameter change formula, we have

$$\frac{\partial x_+}{\partial \mu}(0, 0) = \phi(0, T)\frac{\partial x_+}{\partial \mu}(T, 0) - \int_0^T \phi(0, s)\frac{\partial f}{\partial \mu}(x_*(s)ds, 0),$$

and

$$\frac{\partial x_-}{\partial \mu}(0, 0) = \phi(0, -T)\frac{\partial x_-}{\partial \mu}(-T, 0) - \int_{-T}^0 \phi(0, s)\frac{\partial f}{\partial \mu}(x_*(s), 0)ds.$$

So

$$\begin{aligned}
\frac{d\xi_-}{d\mu}(0) &= \psi_0 \frac{\partial x_-}{\partial \mu}(0, 0) \\
&= \psi(0)(\phi(0, -T)\frac{\partial x_-}{\partial \mu}(-T, 0) + \int_{-T}^0 \phi(0, s)\frac{\partial f}{\partial \mu}(x_*(s), 0)ds) \\
&= \psi(-T)\frac{\partial x_-}{\partial \mu}(-T, 0) + \int_{-T}^0 \psi(s)\frac{\partial f}{\partial \mu}(x_*(s), 0)ds.
\end{aligned}$$

Now, if $t \to \infty$, we obtain

$$\lim_{T \to +\infty} \psi(-T)\frac{\partial x_-}{\partial \mu}(-T, 0) = 0.$$

So

$$\frac{d\xi_-}{d\mu}(0) = \int_{-\infty}^0 \psi(s)\frac{\partial f}{\partial \mu}(x_*(s), 0)ds.$$

Similarly, we get

$$\frac{d\xi_+}{d\mu}(0) = \int_0^{+\infty} \psi(s)\frac{\partial f}{\partial \mu}(x_*(s), 0)ds,$$


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and so

$$s'(0) = \xi'_-(0) - \xi'_+(0) = \int_{-\infty}^{\infty} \psi(s) \frac{\partial f}{\partial \mu}(x_*(s), 0).ds$$

The recent integral is called the Melnikov integral if the function $f$ is dependent on $t$, that is, the right side of the differential equation is $f(x, \mu, t)$, then the value of the Melnikov integral will also depend on $t$.

## 3. **Conclusions**

This article presents a unique perspective on constructing the Melnikov integral, different from what is typically found in textbooks. The problem's assumptions are presented in a way that accounts for both homoclinic and heteroclinic states. By following the method outlined in the article, interested readers can calculate Melnikov integral formula in the time-dependent mode, by considering the time-dependent perturbation on the system.

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**Majid Karimi Amaleh**

Department of Mathematics, University of Hormozgan Bandar Abbas, Iran

Email: karimi@hormozgan.ac.ir