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ON 4-DIMENSIONAL EINSTEINIAN MANIFOLDS WITH PARALLEL NULL DISTRIBUTION

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Abstract. In this paper, we investigate the Einsteinian manifolds with parallel null distribution. For this purpose, we first obtain the equations, which are known as Einstein's equations, that lead to finding the mentioned manifolds and then, we reduce Einstein's equations by using Lie symmetry method. In this method, we first obtain the generators of the symmetry algebra and then calculate the differential Invariants for each of the generators and calculate the group invariant solutions of this equation. We also obtain the optimal system of the one-dimensional sub-algebras of these equations which helps us to have a classification on group invariant solutions by using conjugate mapping.

1. **Introduction**

In the theory of general relativity, the volume of space-time matter can be described by the energy tensor. The fact that the mass of matter in the universe can be considered as a complete fluid in the standard cosmological model led to the hypothesis that it is possible to model the tendency field with appropriate 4-dimensional Lorentzian metrics. This fact made the role of such metrics and the manifolds resulting from it colorful in physics and especially in the theory of relativity [1].

In the meantime, 4-dimensional manifolds that have an null and parallel distribution play an essential role. These manifolds are called Walker manifolds. According to the conditions that must be

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established on the metric of 4-dimensional Walker manifolds, a system of equations is created, which is called Einstein's equations [20]. So, solving the Einstein's equations leads to finding the general metric form of Walker manifolds. These manifolds play a significant role in modeling physical problems and have been studied recently by many mathematicians and physicists [8]. It is worth mentioning that the Walkers' manifolds were first introduced by Walker and was investigated around the years 1950 to 1953 (he studied the manifolds that have a null parallel distribution). In honor of this great mathematician and due to his findings on this field, these manifolds are called Walker manifolds [18, 19].

Einstein's equations play an essential role in making cosmological models. These models express the fact that matter determines the geometry of space-time and, opposite to the movement of matter, it is defined by the space meter tensor [17]. Accordingly, the importance four-dimensional Einstein Walker manifolds as background space for physical models and geometrical as well as solving Einstein's equations to obtain such manifolds are specified. One of the most important and practical methods for analyzing and solving equations is the Lie symmetry method which was first introduced by Marius Sophos Lie in the late 19th century [13]. This method is based on the integration of differential equations and it caused Lie to devote his efforts to the development of the theory of differential equations based on the concept of Lie groups. The Lie groups have many applications in algebraic topology, differential geometry, special functions, numerical analysis, control theory, mechanics classical and quantum relativity. It should be noted that differential equations are the main source of the application of Lie groups in this strings. If we want to briefly talk about the symmetry group of a system of differential equations, we can say that these groups include transformations that transform the solutions of the system. From Lie's point of view, symmetries are geometric transformations that act on the space containing the independent and dependent variables of the system and also act on the solutions of the system with the changes applied to their graph. One of Lie's most important discoveries in the field of differential equations is that he was able to show that the complex nonlinear conditions governing a dynamic system can be locally converted into linear conditions by the infinitesimal derivatives of that system under the generators of its symmetry group, which is important in physics. This led to the use of computers and many software in this field because working with symmetric groups has calculations that follow an algorithmic process [4]. Having the symmetry group of a system of differential equations has many advantages, for example, we can mention the classification of the solutions of the system of differential equations. The basis of this classification is that the solutions that are placed in a category can be converted to each other by some generators of the symmetry group [16].

If we are involved with an ordinary differential equation system, the symmetry group helps us to get the solution by one time integration by reducing the order of the equation to one, and in the case that

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the desired equation is of the first order, the general solution is also obtained. But unfortunately, this is not the case for partial differential equations, that is, the general solution of such equations cannot necessarily be obtained by having a group of symmetries. (Except in the case where the system can be converted into a linear system.) In this situation, only some solutions are obtained, which fall under some subgroups of the symmetry group. These solutions are known as group invariant solutions, which are obtained from the solution of a system that contains a smaller number of independent variables than the original system [5]. In recent years, several equations have been investigated using this method and their group invariant solutions have been obtained. For example, the KdV equation and its different modes, which is one of the important models in the theory of long waves in shallow waters and other physical systems, have been studied in several sources, including [2, 11]. Also, exact solutions of the BBM equation, which is one of the important equations in the modeling of gravity waves with small amplitude, have also been obtained with this method [10, 12].

Our goal in this article is to analyze the symmetric group and obtain group-invariant solutions for Einstein's equations. In addition, we obtain a classification on these solutions based on the optimal system of one-dimensional subalgebras of symmetric algebra. The division of this article is as follows: In the second section, the prerequisites and preliminary definitions and some important results about Riemannian pseudo-geometry and Einsteinian tunnels are stated. In the third section, Lie's symmetry method is stated and the symmetry group of Einstein's equations is obtained. Also, a complete analysis of this group and Lie algebras like it will be presented. In the fourth section, the optimal system of the one-parameter subalgebras of the symmetric algebra of Einstein's equations is presented, and then the reduced equations are obtained using differential derivations. In the end, by solving the reduced equations, the solutions of Einstein's equations are obtained.

2. Main Results

We first prove that the infinitesimal generators of the symmetric Lie algebra of the Einstein's equations and its general form, the second order extension and the characteristic of vector field V are as follows:

$$\begin{aligned} &V_1 = \partial_x, && V_5 = b\partial_c + 2c\partial_a + t\partial_x,\\ &V_2 = \partial_t, && V_6 = c\partial_c + 2b\partial_b + t\partial_t,\\ &V_3 = -c\partial_c + -2b\partial_b + x\partial_x, && V_7 = c\partial_c + b\partial_b + a\partial_a.\\ &V_4 = a\partial_c + 2c\partial_b + x\partial_t, && \end{aligned} \tag{2.1}$$

$$\begin{aligned} V = {} & \xi^1(x,t,a,b,c)\partial_x + \xi^2(x,t,a,b,c)\partial_t + \\ & \phi_1(x,t,a,b,c)\partial_a + \phi_2(x,t,a,b,c)\partial_b + \phi_3(x,t,a,b,c)\partial_c. \end{aligned} \tag{2.2}$$

$$\begin{aligned} V^{(2)} &= V + \phi_1^x \partial_{a_x} + \phi_1^t \partial_{a_t} + \phi_2^x \partial_{b_x} + \phi_2^t \partial_{b_t} + \phi_3^x \partial_{c_x} + \phi_3^t \partial_{c_t} \\ &\quad + \phi_1^{xx} \partial_{a_{xx}} + \phi_2^{xx} \partial_{b_{xx}} + \phi_3^{xx} \partial_{c_{xx}} + \phi_1^{xt} \partial_{a_{xt}} + \phi_2^{xt} \partial_{b_{xt}} \\ &\quad + \phi_3^{xt} \partial_{c_{xt}} + \phi_1^{tt} \partial_{a_{tt}} + \phi_2^{tt} \partial_{b_{tt}} + \phi_3^{tt} \partial_{c_{tt}}. \end{aligned}$$

$$Q_1 = \phi_1 - \xi^1 a_x - \xi^2 a_t, \qquad Q_2 = \phi_2 - \xi^1 b_x - \xi^2 b_t, \qquad Q_3 = \phi_3 - \xi^1 c_x - \xi^2 c_t.$$

Then, by using the invariant theorem, we obtain the following irreducibility conditions:

$$\begin{aligned} & V^{(2)}[a_{xx} - b_{tt}] = 0, && a_{xx} - b_{tt} = 0, \\ & V^{(2)}[c_{xx} + b_{xy}] = 0, && c_{xx} + b_{xy} = 0, \\ & && \vdots \\ & V^{(2)}[-bc_{xt} + 2cb_{xt} + ab_{xx} - {c_x}^2 + b_t c_x - b_x c_t + a_x b_x] = 0, \\ & && -bc_{xt} + 2cb_{xt} + ab_{xx} - {c_x}^2 + b_t c_x - b_x c_t + a_x b_x = 0. \end{aligned} \tag{2.3}$$

Now, by replacing the extension coefficients in (2.3), we get the following determining system:

$$\begin{aligned} & a^3 \xi_a^1 = 0, \quad b^2 \xi_b^1 = 0, \quad a \xi_b^2 = 0, \quad b^2 \xi_b^2 = 0, \quad ac\phi_{2_a} = 0, \quad cb\xi_b^1 = 0, \ldots \\ & abc(-3\xi_t^1 - \phi_{3_b}) + 2c^2 a(-\xi_t^2 + \xi_x^1) + ba^2(\phi_{2_b} - \phi_{3_c} - \xi_x^1 + \xi_t^2) + \\ & \qquad ca^2(-2\xi_x^2 + \phi_{2_c}) + \phi_1(ab - 2c^2) - a^2\phi_2 + 2ca\phi_3 + 4c^3\xi_t^1 = 0. \end{aligned} \tag{2.4}$$

In the following theorem, by solving the above system of PDEs, the vector field coefficients (2.2) is obtained.

Theorem 2.1. *The Lie algebra corresponding to the symmetry group of the Einstein's system of equations is produced by the vector field 2.2, whose coefficients are the following functions:*

$$\begin{aligned} & \phi_1 = k_7 a + 2k_1 c, && \xi_1 = k_2 + k_1 t + k_3 x, \\ & \phi_2 = 2k_6 c + (k_7 - 2k_3 + 2k_4)b, && \xi_2 = k_5 + k_4 t + k_6 x, \\ & \phi_3 = k_1 b + k_6 a + (-k_3 + k_4 + k_7)c, \end{aligned} \tag{2.5}$$

where k_is, $i = 1, \ldots, 7$, are arbitrary constants. The one-parameter group related to the vector fields 2.1, which is denoted by $g_k(\varepsilon)$ for V_k, $k = 1, \ldots, 6$, is defined as follows:

$$\begin{aligned} & g_1(\varepsilon) : (x,t,a,b,c) \longmapsto (x+\varepsilon, t, a, b, c), \qquad g_2(\varepsilon) : (x,t,a,b,c) \longmapsto (x, t+\varepsilon, a, b, c), \\ & g_3(\varepsilon)(x,t,a,b,c) \longmapsto (e^{\varepsilon} x, t, a, e^{-2\varepsilon} b, e^{-\varepsilon} c), \\ & g_4(\varepsilon) : (x,t,a,b,c) \longmapsto (x, t+\varepsilon x, a, b + 2\varepsilon c + \varepsilon^2 a, c + \varepsilon a), \\ & g_5(\varepsilon) : (x,t,a,b,c) \longmapsto (x + \varepsilon t, t, a + 2\varepsilon c + \varepsilon^2 b, b, c + \varepsilon b), \\ & g_6(\varepsilon) : (x,t,a,b,c) \longmapsto (x, e^{\varepsilon} t, a, e^{2\varepsilon} b, e^{\varepsilon} c), \quad g_7(\varepsilon) : (x,t,a,b,c) \longmapsto (x, t, e^{\varepsilon} a, e^{\varepsilon} b, e^{\varepsilon} c). \end{aligned} \tag{2.6}$$

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3. Conclusions

As one of the most important equations in mathematics and physics, we analyzed the Einstein's equations, by using the Lie symmetry method. We first obtained the symmetric group of these equations and the optimal system of one-parameter subalgebras. Then, using differential invariants, we reduced the equations and obtained the group invariants solutions, which lead to the new solutions of the Einstein's equations and to the new Einsteinian Walker manifolds.

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