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STAR COMPRESSED ZERO DIVISORS GRAPH AND PARTITIONS OF VECTOR SPACES

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Abstract. Let R be a commutaive ring and $Zd(R)$ be the set of zero divisors of R. Define an equivalence relation $\sim$ on $Zd(R)$ as follows: $x \sim y$ if and only if $ann(x) = ann(y)$. The graph $\Gamma_E(R)$ is a graph associated to R whose vertices are the classes of elements in $Zd(R)^* = Zd(R)\backslash\{0\}$, and two distinct classes $[x] \neq [y]$ are joined by an edge if and only if $xy = 0$. We show that if R is a local ring and $\Gamma_E(R)$ is a star graph with at least four elements then $m/Soc(R)$ has a partition of vector spaces where m is the maximal ideal of R. Also, We construct from a special partition of vector spaces, a ring whose associated graph is a star graph.

1. Introduction

The compressed zero divisor graph or the graph of equivalence classes of zero divisors of a ring R is denoted by $\Gamma_E(R)$, and is defined in [20]. Let $Zd(R)$ denotes the set of zero divisors of a ring R and $Zd(R)^* = Zd(R)\backslash\{0\}$. Define a relation $\sim$ on $Zd(R)$ as follows[14]: $x \sim y$ if and only if $ann(x) = ann(y)$. It is easily seen that $\sim$ is an equivalence relation. The graph $\Gamma_E(R)$ is a graph associated to R whose vertices are the classes of elements in $Zd(R)^*$, and two distinct classes $[x] \neq [y]$ are joined by an edge if and only if $xy = 0$. Another interpretation of $\Gamma_E(R)$ is as follows: The vertices are the elements of $\mathcal{J} = \{ann(a) : a \in Zd(R)^*\}$ and two distinct elements $ann(x)$ and $ann(y)$ are adjacent if and only if $xy = 0$. In [20], some necessary conditions are obtained for a ring R such

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that $\Gamma_E(R)$ is a star graph. For example, If $\Gamma_E(R)$ is a star graph with at least four vertices then $|Ass(R)| = 1$ and $Char(R) = 2, 4, 8$. In [9] a method for constructing star compressed zero divisor graph is obtained. They used a quotient of a symmetric algebra of a vector space whose relations come from a special partition of that vector space. But it seems that the authors were not aware of partition of vector spaces. By analyzing the proofs in [20] and [9], we see that the star graphs give a partition of a vector space and conversely some partitions give a star graph. An interesting problem about star compressed zero divisor graph is the size of them. For example is there any ring whose compressed zero divisor graph be a star graph with 36 vertices?

2. Main Results

Theorem 2.1. *Let R be a ring such that $\Gamma_E(R)$ is a star graph with at least four vertices. Let $[y]$ be the unique vertex with maximal degree and $K = [y]\bigcup\{0\}$. Then R satisfies the following properties:*

(1) *$Ass(R) = \{P\}$ where $ann(y) = P$. Also $ann(P) = K$. In particular K is an ideal of R and $Zd(R) = P$.*
(2) *$P^3 = 0$.*
(3) *If $ann(x_0) = K$ then $x_0^3 = 0$ and $[x_0 + y] = [x_0]$.*
(4) *If $J = \{x \in R : K \subsetneq ann(x)\}$ then J is an ideal of R and $ann(x) = [x]\bigcup K$ for each $x \in J\backslash K$. Also $[x + y] = [x]$ for each $x \in J\backslash K$.*
(5) *$Char(R) = 2, 4, 8$.*

Corollary 2.2. *Let (R, m) be a local Artinian ring such that $\Gamma_E(R)$ is a star graph with at least four vertices. Let $[y]$ be the unique vertex with maximal degree and $K = [y]\bigcup\{0\}$. If $J = \{x \in R : K \subsetneq ann(x)\}$ then $\{ann(x)/K : x \in J\backslash(K)\}$ is a partition of $R/m-$vector space J/K. Also $Soc(R) = ann(m) = K$.*

Theorem 2.3. *Let $V = V_1 \bigoplus \cdots \bigoplus V_t$ be an $n-$dimensional vector space over $\mathbb{F}_2$ and $dim(V_i) = n_i$. Assume $\{X_{i,k} : 1 \leq k \leq n_i\}$ is a basis of V_i. Let $S = \mathbb{F}_2[X_{i,k} : 1 \leq i \leq t, 1 \leq k \leq n_i]$ be a polynomial ring over n indeterminates. Let $I = \langle V_i^2, V^3\rangle$ and $R = \frac{S}{I}$. Then $\Gamma_E(R)$ is a star graph with $2^n - (2^{n_1} + \cdots + 2^{n_t}) + 2t$ vertices.*

3. Summary of Proofs/Conclusions

In this article we show that every star compressed zero divisor graph correspond to a partition of vector spaces. Conversely, we construct from a special vector space partition a star compressed zero divisor graph.

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