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SOME PASCAL-LIKE TRIANGLES

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Abstract. In this article, we delve into the fascinating world of numerical triangles similar to Khayyam-Pascal triangle. Our focus is on triangles that are comprised of natural numbers. Along with a combinatorial interpretation, we also propose an algebraic interpretation for the elements in most cases. We explore in details the triangle of coefficients of Chebyshev polynomials (Chebyshev triangle). Through our analysis, we derive a recursive relation for its elements. This finding sheds new light on the properties of this intriguing numerical construction. To further enhance our understanding of these triangles, we also present new images related to the Catalan, Bell, and Chebyshev triangles. These images provide a clearer visualization of the numerical triangle construction. Overall, this article offers a comprehensive exploration of numerical triangles similar to Khayyam-Pascal triangle and examine some of their properties and relationships for better understanding of these constructions.

1. Introduction

The history of the Pascal's triangle dates back to the 8th century AD, but the synchronization of Pascal's work with the scientific revolution after the Middle Ages and using the triangle for solving many problems in combinatorial analysis in Pascal's work led to the recognition of this numerical triangle as Pascal's triangle in the world of mathematics. Although the triangle had been known as early as in

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ancient India, and later in Persia, China and Europe in the Middle Ages by a number of scientists before Pascal. In Persia, Khayyam and Karaji referred the triangle and binomial coefficients in the 11th century AD [5, 6]. Due to Khayyam's advancements in calculating binomial coefficients, the triangle is known as Khayyam-Pascal triangle in Iran.

Considering the numerous applications of the Khayyam-Pascal triangle and the fantastic patterns in it, various generalizations have been proposed for it. In this article, which is somewhat a continuation of article [1], we introduce some of the most famous numerical triangles that consist of natural numbers and have a structure similar to Khayyam-Pascal triangle. In some places, we have made changes in arguments or shapes. Two notable changes have been made in illustrating Bell's Triangle and Catalan's Triangle. The proof of the relationship between elements in triangle of coefficients of Chebyshev polynomials (Chebyshev triangle) is also relatively new.

2. Discussion/Main Results

2.1. **Khayyam-Pascal triangle.** Khayyam-Pascal triangle is an infinite array of numbers which is constructed by beginning with the number 1 and with 1's running down the two sides of triangle. Each new number lies between two numbers and below them is the sum of the two numbers above it. Repeating this algorithm, we can create the triangle shown in Figure 1. The entries in nth row of the triangle are the coefficients of the binomial expansion $(x + y)^n$. By the Newton's binomial theorem

$$(x + y)^n = \Sigma_{k=1}^{n} \binom{n}{k} x^k y^{n-k}.$$

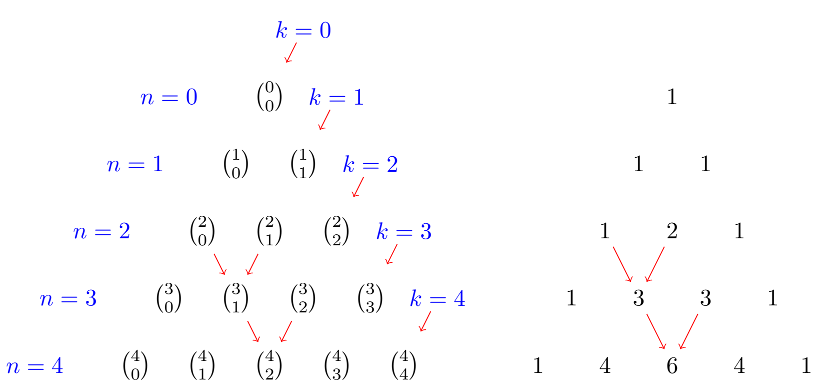

Figure 1. Khayyam-Pascal triangle

2.2. $(2, 1)$**-Pascal triangle or Lucas triangle.** The $(2, 1)$-Pascal triangle or Lucas triangle is constructed as like as the Khayyam-Pascal triangle in which its leftmost entries initialized to 2 and its rightmost entries (except the first row) initialized to 1. The inside numbers are sums of the two adjacent values of the preceding row (Figure 2). The entries in nth row of the $(2, 1)$-Pascal are the coefficients of the binomial expansion $(2x + y)(x + y)^{n-1}$.

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The (2, 1)-Pascal triangle is not symmetrical like as the Khayyam-Pascal triangle. However, similar to the Khayyam-Pascal triangle, it exhibits numerous patterns. For example, square numbers are located on diagonal $k = 2$. Additionally, it possesses the Hockey Stick property. More precisely, if $T_{(2,1)}(n, k)$ represents the element in the nth row and kth diagonal of the (2, 1)-Pascal triangle, then

$$T_{(2,1)}(0, k)+T_{(2,1)}(1, k)+\cdots+T_{(2,1)}(n, k) = T_{(2,1)}(n+1, k+1).$$

For more details see [8]. Also see [7] for a fantastic property of (2,1)-Pascal triangle.

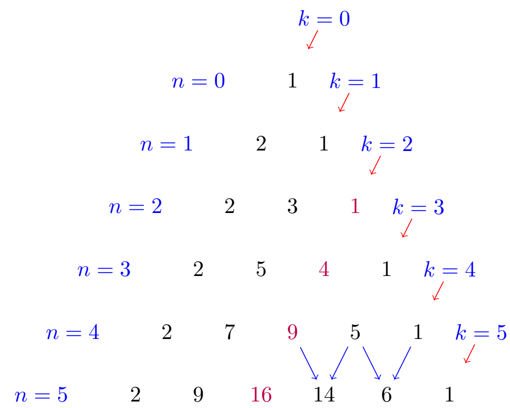

Figure 2. (2,1)-Pascal triangle

2.3. **Pascal's pyramid.** Pascal's pyramid is the three-dimensional analog of the two-dimensional Khayyam-Pascal triangle. The elements of the nth row of the Pascal's pyramid are the coefficients of the trinomial expansion $(x + y + z)^n$. By the trinomial expansion,

$$(x + y + z)^n = \sum_{0\leq k_i\leq n, k_1+k_2+k_3=n} \binom{n}{k_1, k_2, k_3} x^{k_1} y^{k_2} z^{k_3}$$

where $\binom{n}{k_1, k_2, k_3} = \frac{n!}{k_1!k_2!k_3!}$. For $n \geq 3$, and $i, j, k \geq 1$ these coefficients satisfy the recurrence relation

$$\binom{n}{i, j, k} = \binom{n-1}{i-1, j, k} + \binom{n-1}{i, j-1, k} + \binom{n-1}{i, j, k-1},$$

which leads to the formation of Pascal's pyramid shown in Figure 3. For more details see [2, 9].

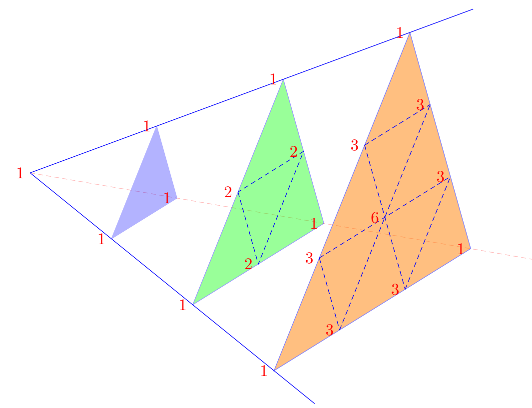

Figure 3. Pascal's pyramid

2.4. **Triangle of trinomial coefficients.** Coefficients of the expansion $(1+x+x^2)^n$ are called trinomial coefficients. If theses coefficients are shown by $\binom{n}{k}_2$, then

$$(1 + x + x^2)^n = \sum_{k=-n}^{n} \binom{n}{k}_2 x^{n+k} = \sum_{k=0}^{2n} \binom{n}{k-n}_2 x^k.$$

Note that for any $n \geq 0$ and $-n \leq k \leq n$, $\binom{n}{k}_2 = \binom{n}{-k}_2$. The numbers on the nth row of triangle of trinomial coefficients are $\binom{n}{-n}_2, \binom{n}{-n+1}_2, \ldots, \binom{n}{n}_2$. For more details see [3, 10].

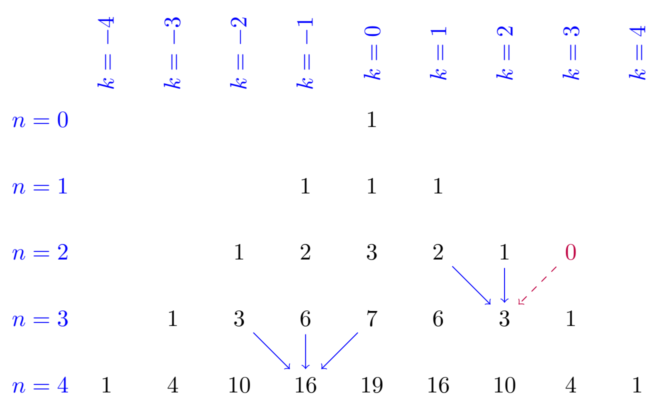

Figure 4. triangle of trinomial coefficients

2.5. **Stirling triangles.** Stirling numbers are defined in various combinatorial and algebraic interpretations. There are three kind of Stirling numbers. The Stirling numbers of the first kind, the Stirling numbers of the second kind, and the Lah numbers which sometimes referred to as the Stirling numbers of the third kind. A common algebraic property of all three kinds is that they describe coefficients relating three different sequences of polynomials that frequently arise in combinatorics. Moreover, all three can be defined as the number of partitions of n elements into k non-empty subsets, where each subset is endowed with a certain kind of order. The falling factorial is defined as the polynomial $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ while the rising factorial is $\langle x\rangle_n = x(x+1)(x+2)\cdots(x+n-1)$.

The unsigned Stirling numbers of the first kind are denoted by $\left[{n \atop k}\right]$ and algebraically are defined as the coefficients of the powers of x in the expansion $\langle x\rangle_n$, i.e., $\langle x\rangle_n = \sum_{k=0}^{n}\left[{n \atop k}\right]x^k$. According to this definition, it can be shown that $\left[{0 \atop 0}\right] = 1$, $\left[{1 \atop 1}\right] = 1$, $\left[{n \atop n}\right] = 1$, and $\left[{n \atop 0}\right] = 0$ for each natural number n. Furthermore, $\left[{n+1 \atop k}\right] = \left[{n \atop k-1}\right] + n\left[{n \atop k}\right]$ for each $1 \leq k \leq n$. Hence, the Stirling triangle of the first kind constructed as Figure 5. It can be show that the number of permutations on n elements with k cycles is also equal to the unsigned Stirling number $\left[{n \atop k}\right]$. See [11] for more details.

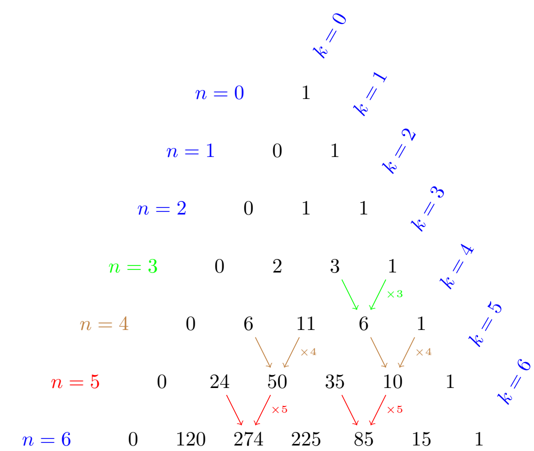

Figure 5. triangle of the Stirling numbers of the first kind

Algebraically, the Stirling numbers of the second kind which are denoted by $\left\{{n \atop k}\right\}$ are defined as the coefficients of the falling factorials in the expansion of x^n, i.e., $x^n = \sum_{k=0}^{n}\left\{{n \atop k}\right\}(x)_k$. Combinatorially, Stirling numbers of the second kind $\left\{{n \atop k}\right\}$ counts the number of ways in which n distinguishable objects can be partitioned into k indistinguishable nonempty subsets. Accordingly, one could established that $\left\{{n \atop 1}\right\} = 1$, $\left\{{n \atop n}\right\} = 1$ for each $n \geq 0$. Furthermore, $\left\{{n \atop 0}\right\} = 0$ for each $n \geq 1$ and $\left\{{n+1 \atop k}\right\} = \left\{{n \atop k-1}\right\} + k\left\{{n \atop k}\right\}$ for each $1 \leq k < n$. See [11, 12] for more details.

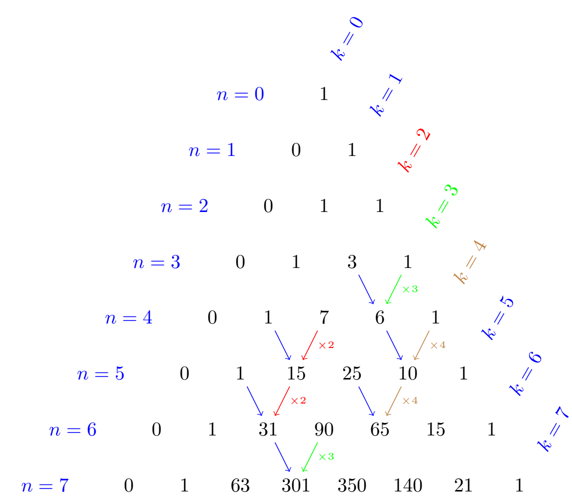

Figure 6. triangle of Stirling numbers of the second kind

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Lah numbers or the Stirling numbers of the third kind are denoted by $L(n,k)$ and defined as the coefficients of the falling factorials in the expansion of $\langle x\rangle_n$, i.e., $\langle x\rangle_n = \sum_{k=1}^{n} L(n,k)(x)_k$. Lah numbers count the number of ways that a set of n elements can be partitioned into k nonempty linearly ordered subsets. They satisfies the recursive relation

$$L(n+1,k) = (n+k)L(n,k) + L(n,k-1).$$

Triangle of the Lah numbers is show in Figure 7. For more details see [11, 13].

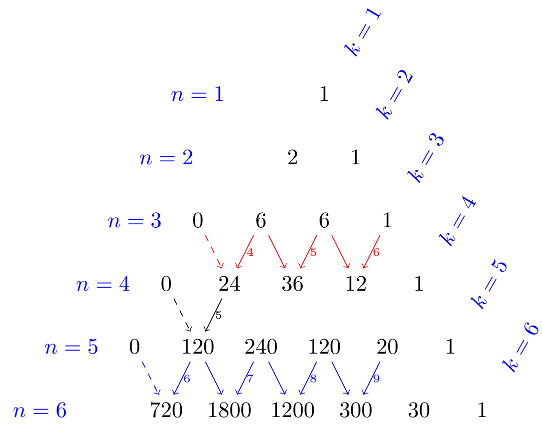

Figure 7. triangle of Lah numbers

2.6. **Triangle of Eulerian numbers (Euler triangle).** The Eulerian number $\left\langle {n \atop m} \right\rangle$ is the number of permutations of the numbers $\{1,\cdots,n\}$ in which exactly m elements are greater than previous element. For example, there are 4 permutations of the numbers $\{1,2,3\}$ in which exactly 1 element is greater than the previous elements. 132 in which 13 is ascending, 213 in which 13 is ascending, 231 in which 23 is ascending, and 312 in which 12 is ascending. Note that 321 has no ascending sequence and 123 has two ascending sequences. Therefore, $\left\langle {3 \atop 1} \right\rangle = 4$. Eulerian numbers satisfies the recursive relation $\left\langle {n+1 \atop k} \right\rangle = (k+1)\left\langle {n \atop k} \right\rangle + (n+1-k)\left\langle {n \atop k-1} \right\rangle$. Hence, they formed a numerical triangle pictured in Figure 8. Algebraically, Eulerian Numbers are the coefficients of the binomial coefficients in the expansion $x^n = \left\langle {n \atop 0} \right\rangle \binom{x}{n} + \left\langle {n \atop 1} \right\rangle \binom{x+1}{n} + \cdots + \left\langle {n \atop k} \right\rangle \binom{x+k}{n} + \cdots + \left\langle {n \atop n-1} \right\rangle \binom{x+n-1}{n}$ of x^n. See [12, 14] for more details.

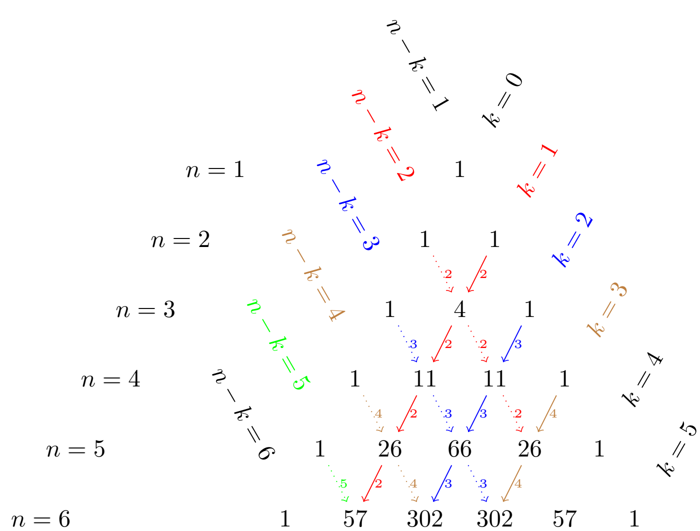

Figure 8. Euler triangle

2.7. **Catalan's triangle.** The Catalan numbers are denoted by C_n, $n \geq 0$, and are given by the explicit formula $C_n = \frac{1}{n+1}\binom{2n}{n}$. Combinatorially, C_n counts the number of expressions containing n pairs of parentheses which are correctly matched. C_n also count the number of Dyck words of length $2n$. A Dyck word is a string consisting of n X's and n Y's such that no initial segment of the string has more Y's than X's. For example, the only Dyck words of length 2 is XY, and Dyck words of length 4 are XXYY and XYXY. A generalization of the latest interpretation of Catalan numbers, defined $C(n,k)$ as the number of strings consisting of n X's and k Y's such that no initial segment of the string has more Y's than X's.

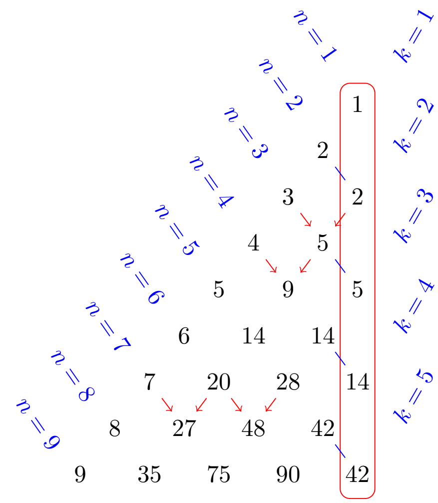

Figure 9. Catalan's triangle

$C(n,k)$ satisfies the recursive relations $C(n,0) = 1$ for $n \geq 0$, $C(n,1) = n$ for $n \geq 1$, $C_{n+1,k} = C_{n+1,k-1} + C_{n,k}$ for $1 < k < n+1$, and $C_{n+1,n+1} = C_{n+1,n}$ for $n \geq 1$. Thus they formed the numerical triangle in Figure 9. Note that the Catalan number C_n is equal to $C(n,n)$. For more details see [2, 12, 15, 16, 17].

2.8. **Narayana Triangle.** Another interpretation of Catalan number C_n says that C_n counts the number of expressions containing n pairs of parentheses which are correctly matched. For example, $C_3 = 5$. Indeed, the only correctly matched parentheses of length three are ( )( )( ) , ( )(( )) , (( ))( ) , ((( ))) , (( )( )).

Narayana number $N_{n,k}$ generalize this interpretation. Indeed, $N_{n,k}$ count the number of words containing n pairs of parentheses, which are correctly matched and which contain k distinct nestings. For instance, $N_{4,2} = 6$, since with four pairs of parentheses, six sequences can be created which each contain two occurrences the sub-pattern ().

( )((( ))), (( ))(( )), (( )(( ))), ((( ))( )), ((( )( ))), ((( )))( )

One can established that $N_{n,k} = \binom{n-1}{k}\binom{n+1}{k+1} - \binom{n}{k+1}\binom{n}{k}$. So, Narayana numbers formed the triangle in Figure 10. Yellow numbers are Khayyam-Pascal triangle and the Narayana numbers are extracted from 2×2 minors of Khayyam-Pascal triangle. More details on [14, 18, 19].

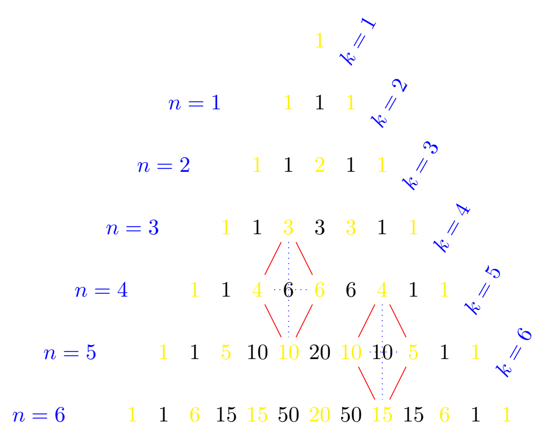

Figure 10. Narayana triangle

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2.9. **Bernoulli's Triangle.** Bernoulli's triangle is an array of partial sums of the binomial coefficients. For any non-negative integer n and for any integer m included between 0 and n, the component in row n and column m of Bernoulli's triangle is $B_{n,m} = \sum_{k=0}^{m} \binom{n}{k}$. Similarly to Khayyam-Pascal triangle, each component of Bernoulli's triangle is the sum of two components of the previous row, except for the last number of each row, which is double the last number of the previous row. So, $B_{n+1,m} = B_{n,m} + B_{n,m-1}$ for $m < n+1$ and $B_{n+1,n+1} = 2B_{n,n}$ (Figure 11).

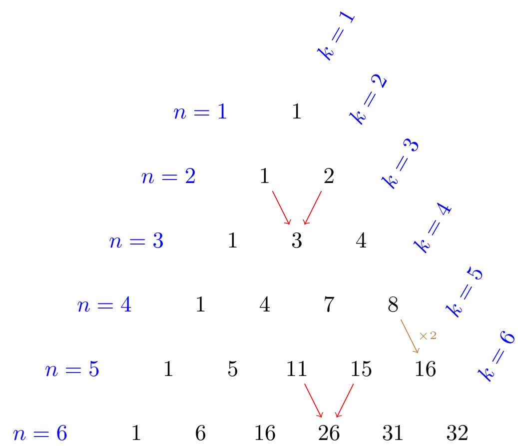

Figure 11. Bernoulli's triangle

2.10. **Bell triangle.** Bell numbers is an array of total sums of the Stirling numbers of the second kind, i.e. for any non-negative integer n, $B(n) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\}$. The components of the Bell triangle are obtained by beginning the first diagonal with the number one, and begin all other diagonals with the last number in first diagonal, and filling out the other elements of each row by adding the elements in the preceding row. Combinatorially, Bell numbers count the number of ways of partitioning a finite set into subsets, or equivalently the number of equivalence relations on a finite set. See also [22, 23, 24, 12, 16, 25].

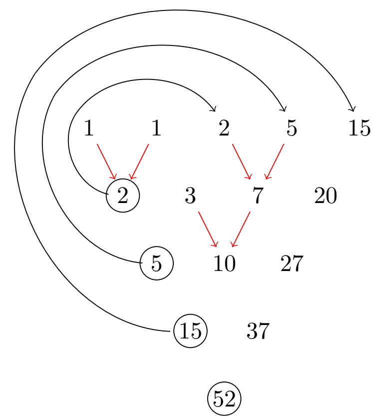

Figure 12. Bell triangle

2.11. **Triangle of the Coefficients of Chebyshev Polynomials.** Chebyshev polynomials are defined through identities $T_n(\cos\theta) = \cos(n\theta)$ and $U_{n-1}(\cos\theta) = \frac{\sin(n\theta)}{\sin\theta}$. $\{T_n(x)\}_{n=0}^{\infty}$ are called the Chebyshev polynomials of the first kind and $\{U_n(x)\}_{n=0}^{\infty}$ are called the Chebyshev polynomials of the second kind. Chebyshev polynomials satisfies several recursive relations. We find a new recursive relation for the coefficients of Chebyshev polynomials which leads to the triangle for the coefficients of Chebyshev Polynomials shown if Figure 13.

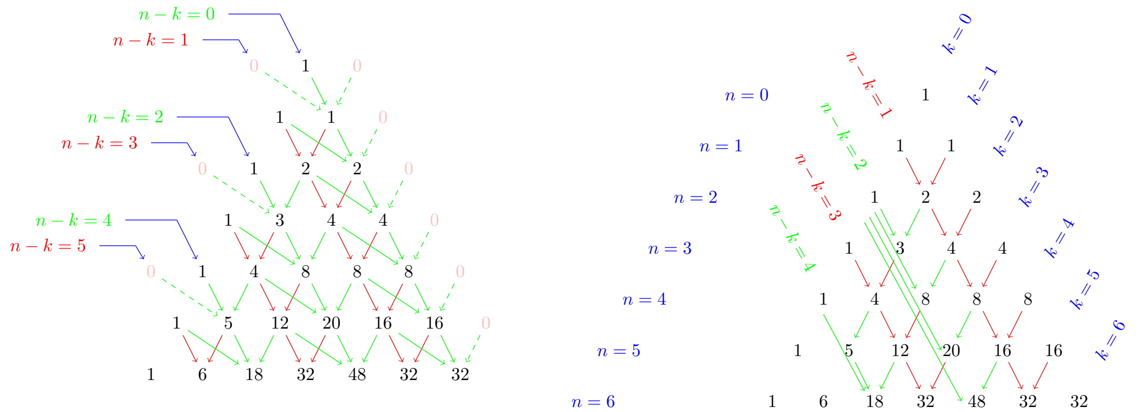

Figure 13. triangle of the coefficients of Chebyshev Polynomials

3. Summary of Proofs

In this section we state a new recursive relation around Chebyshev polynomials which leads to a triangle for the coefficients of Chebyshev Polynomials shown if Figure 13. Following identities $T_n(\cos\theta) = \cos(n\theta)$ and $U_{n-1}(\cos\theta) = \frac{\sin(n\theta)}{\sin\theta}$ and using De Moivre's formula one could established that

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= 1x & U_0(x) &= 1 \\ T_2(x) &= -1 + 2x^2 & U_1(x) &= 2x \\ T_3(x) &= -3x + 4x^3 & U_2(x) &= -1 + 4x^2 \\ T_4(x) &= 1 - 8x^2 + 8x^4 & U_3(x) &= -4x + 8x^3 \end{aligned}$$

1
1 1
1 2 2
1 3 4 4
1 4 8 8 8

If we write $T_n(x) = a_{n,n}x^n - a_{n,n-2}x^{n-2} + \cdots$ and $U_{n-1}(x) = a_{n,n-1}x^{n-1} - a_{n,n-3}x^{n-3} + \cdots$ then we could obtain recursive relations $a_{n,k} = a_{n-1,k-1} + a_{n-1,k}$ for $k > 0$ and odd values of $n-k$, and $a_{n,k} = a_{n-1,k-1} + a_{n-1,k} + a_{n-1,k+1}$ for $k > 0$ and even values of $n-k$. This fact leads to the following triangle for the unsigned coefficients of Chebyshev Polynomials. See [20] and [21] also.

References

[1] S. M. A. khatami, Beauties of the Khayyam-Pascal triangle, *Journal of Mathematics and Society*, **5** (2020) 75–92. [In Persian]

[2] M. Mirzavaziri, *Count Countables*, Ahang Ghalam publisher, 2011. [In Persian]

[3] A. Sabetian, Khayyam-Pascal triangle and some generalizations, *Roshd Mathematics Education Journal*, **42** (1995) 54–57. [In Persian]

http://dx.doi.org/10.22108/msci.2023.137812.1578

[4] J. Behboodian, M. Bayat and H. Teimoori Faal, *Khayyam-Pascal numerical triangle and similar triangles*, Sharif University of Technology Publication, 2007. [In Persian]

[5] J. L. Coolidge, The story of the binomial theorem, *Amer. Math. Monthly*, **56** (1949) 147–157.

[6] R. Rashed, *The development of Arabic mathematics: between arithmetic and algebra*, (Translated from the 1984 French original by A. F. W. Armstrong), **156**, Kluwer Academic Publishers, Dordrecht, 1994.

[7] N. A. Draim and M. Bicknell, Sums of n-th powers of roots of a given quadratic equation, *Fibonacci Quart.*, **4** (1966) 170–178.

[8] M. Senn, *(1,2)-Pascal triangle - OeisWiki*, 2016.

[9] H. Belbachir, A. Mehdaoui and L. Szalay, Diagonal sums in Pascal pyramid, *J. Combin. Theory Ser. A*, **165** (2019) 106–116.

[10] G. E. Andrews, Euler's "exemplum memorabile inductionis fallacis" and q-trinomial coefficients, *J. Amer. Math. Soc.*, **3** (1990) 653–669.

[11] T. Mansour and M. Schork, *Commutation relations, normal ordering, and Stirling numbers*, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2016.

[12] R. L. Graham, D. E. Knuth and O. Patashnik. *Concrete mathematics. A foundation for computer science*, Second edition. Addison-Wesley Publishing Company, Reading, MA, 1994.

[13] S. Daboul, J. Mangaldan, M. Z. Spivey and P. J. Taylor, The Lah numbers and the nth derivative of $e^{1/x}$, *Math. Mag.*, **86** (2013) 39–47.

[14] T. Kyle Petersen, *Eulerian numbers*, Birkhäuser Advanced Texts, Birkhäuser/Springer, New York, 2015.

[15] D. F. Bailey, Counting Arrangements of 1's and -1's, *Math. Mag.*, **69** (1996) 128–131.

[16] H. S. Wilf, *generatingfunctionology*, Third edition. A. K. Peters, Ltd., Wellesley, MA, 2006.

[17] W. Eplett, A note about the catalan triangle, *Discrete Math.*, **25** (1979) 289–291.

[18] N. Dershowitz and S. Zaks, Ordered trees and noncrossing partitions, *Discrete math.*, **62** (1986) 215–218.

[19] E. Miller and V. Reiner, *Geometric combinatorics*, AMS, 2007.

[20] W. Lang, *Triangle of coefficients of Chebyshev's $T(n, x)$ polynomials (powers of x in increasing order)*, OeisWiki, 2013.

[21] A. Macdougall, A Pascal-like Triangle for Coefficients of Chebyshev Polynomials, *The Mathematical Gazette*, **83** (1999) 276–280.

[22] G. Dobinski, Summirung der reihe $\sum \frac{n^m}{m!}$ fur $m = 1, 2, 3, 4, 5, \ldots$, *Grunert's Archiv*, **61** (1877) 333–336.

[23] C. S. Peirce, On the algebra of logic, *Amer. J. Math.*, **3** (1880) 15–57.

[24] A. C. Aitken, A problem in combinations, *Edinburgh Mathematical Notes*, **28** (1933) xviii-xxiii.

[25] J. H. Conway and R. K. Guy. *The book of numbers*, Copernicus, New York, 1996.

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