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EQUALIZING IDEAL FOR INTEGER-VALUED POLYNOMIALS OVER THE UPPER TRIANGULAR MATRIX RING

ALI REZA NAGHIPOUR

Abstract. Let D be an integral domain and I be an ideal of the upper trangular matrix ring $T_n(D)$. In this paper, we study the equalizing ideal

$$q_I = \{A \in T_n(D) | f(A) - f(0) \in I, \forall f \in \mathrm{Int}(T_n(D))\}.$$

of the integer-valued polynomials over $T_n(D)$.

1. Introduction

Let $\mathbb{Z}$ and $\mathbb{Q}$ denote the set of integer and rational numbers, respectively. The classical ring of the integer-valued polynomials (over $\mathbb{Z}$) is defined by

$$\mathrm{Int}(\mathbb{Z}) := \{f \in \mathbb{Q}[x] | f(\mathbb{Z}) \subseteq \mathbb{Z}\}.$$

This ring has many interesting properties and has been extensively investigated. For a generalization, let D be an integral domain and let K be the field of fractions of D. Then, the ring of integer-valued polynomials over D is defined by

$$\mathrm{Int}(D) := \{f \in K[X] | f(D) \subseteq D\}.$$

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The first systematic study of the algebraic properties of $\mathrm{Int}(D)$ was done in consecutive 1919 papers of Ostrowski [8] and Polya [9] of the same title.

Let R be a ring and $f(x)$ and $g(x)$ be two elements of $R[x]$. Then $(fg)(x)$ denotes the product of $f(x)$ and $g(x)$ in $R[x]$. If R is not a commutative ring and $\alpha \in R$, then $(fg)(\alpha)$ need not equal $f(\alpha)g(\alpha)$. In this case if $f(x) = \sum_{i=1}^{n} a_i x^i$, then we may express

$$(*) \qquad (fg)(X) = \sum_{i=1}^{n} a_i g(X) X^i.$$

Let S be a non empty set and let $M_n(S)$ and $T_n(S)$ denote the set of $n \times n$ matrices and upper triangular matrices with entries from the set S, respectively. Let D be an integral domain. In 2012, Werner showed that the set

$$\mathrm{Int}(M_n(D)) \quad := \quad \{f \in M_n(K)[X] | f(M_n(D)) \subseteq M_n(D)\}$$

with ordinary addition and multiplication $(*)$ is a ring [12, Corollary 1.3]. In 2017, Frisch proved that

$$\mathrm{Int}(T_n(D)) \quad := \quad \{f \in T_n(K)[X] | f(T_n(D)) \subseteq T_n(D)\}$$

is a ring [3, Theorem 5.4]. Recently, integer-valued polynomials have been considered over some noncommutative rings (see for examples [3, 5, 6, 10]).

In [1], Cahen and Chabert introduced the notion of equalizing ideal for a maximal ideal of D. In 2015, Cahen and Rissner [2] generalized this notion for an arbitrary ideal of the domain D. The authors in [5], introduced and studied the equalizing ideal of every ideal $M_n(\mathfrak{a})$ in $\mathrm{Int}(M_n(D))$, where $\mathfrak{a}$ is an ideal of D.

We generalize the notion of equalizing ideal in the integer-valued polynomial ring $\mathrm{Int}(T_n(D))$. In fact, for an arbitrary ideal I of $T_n(D)$, we define

$$q_I = \{A \in T_n(D) | f(A) - f(0) \in I, \forall f \in \mathrm{Int}(T_n(D))\}.$$

If I is an ideal of $T_n(D)$ and $A \in T_n(D)$, we set

$$\mathcal{J}_{I,A} := \{f \in \mathrm{Int}(T_n(D)) | f(A) \in I\}.$$

In this paper, we study the equaizing ideal q_I and find some relations between q_I and $\mathcal{J}_{I,A}$.

2. Main Results

The main results of this paper are stated as follows:

Theorem 2.1. *Let I be an ideal of $T_n(D)$ and let $A_1, A_2 \in T_n(D)$. Then the following statments hold.*

(1) *q_I is an ideal of $T_n(D)$.*

(2) *$q_I \subseteq I$.*

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(3) *If* $A_1 - A_2 \in q_I$, *then* $\mathcal{J}_{I,A_1} = \mathcal{J}_{I,A_2}$.

Theorem 2.2. *Let* $n > 1$ *and let* $f(X) = B_kX^k + \cdots + B_1X + B_0 \in \mathrm{Int}(T_n(D))$. *Then, the following statments hold.*

(1) $B_0 \in T_n(D)$.
(2) $(B_1)_{ij} \in D$ *for all* $1 \leq i \leq j \leq n-1$.

The following corollary, which is one of the mail results of this paper, immediatelly follows.

Corollary 2.3. *Let* $\mathfrak{a}$ *be an ideal of* D. *Then, we have*

$$T_n(q_{\mathfrak{a}}) \subseteq q_{T_n(\mathfrak{a})}.$$

Let R be a commutative ring with identity and let I be an ideal of $T_n(R)$. By [4], there are ideals I_{ij} $(i \leq j)$ of R such that

$$I = \begin{bmatrix} I_{11} & I_{12} & \dots & I_{1n} \\ 0 & I_{22} & \dots & I_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & I_{nn} \end{bmatrix},$$

where $I_{ii} \subseteq I_{i(i+1)} \subseteq \cdots \subseteq I_{in}$ and $I_{ii} \subseteq I_{(i-1)i} \subseteq \cdots \subseteq I_{1i}$ for all $1 \leq i \leq n$. By this notation, we have the following theorem.

Theorem 2.4. *Let* I *be an ideal of* $T_n(D)$. *Then, we have*

$$q_I = \begin{bmatrix} q_{I_{11}} & I_{12} & \dots & I_{1n} \\ 0 & q_{I_{22}} & \dots & I_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & q_{I_{nn}} \end{bmatrix}.$$

Another main result of this paper is the following corollary.

Corollary 2.5. *Let* I *be a nonzero ideal of ideal of* $T_n(D)$. *Then* $q_I \neq 0$.

3. Conclusions

Let $\mathfrak{a}$ be an ideal of D. It is not always easy to compute the equalizing ideal $q_{T_n(\mathfrak{a}))}$ (or $q_{\mathfrak{a}}$). However, it is known that, for every nonzero ideal $\mathfrak{a}$ of a one-dimensional, Noetherian, local domain D with finite residue field, the residue ring $\frac{D}{\mathfrak{a}}$ is finite. In this paper, we prove that the set of distinct ideals of the form $\mathcal{J}_{T_n(\mathfrak{a}),A}$ is finite for one-dimensional, Noetherian, local domain D with finite residue field ($A \in T_n(D)$ and $\mathfrak{a}$ be an ideal of D). Let D be a local domain with maximal ideal $\mathfrak{m}$ and $a \in D$. As a consequence, we have that, if D is a Noetherian local one-dimensional domain with finite residue field, which is not unibranched, then the set of distinct ideals $\mathcal{J}_{\mathfrak{m},a}$ of $\mathrm{Int}(D)$ above the maximal ideal $\mathfrak{m}$ of D is finite (see [1, Proposition V.3.10]).

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Ali Reza Naghipour

Department of Pure Mathematics, Faculty of Mathematical Sciences, Shahrekord University, P. O. Box 115, Shahrekord, Iran

`Email: naghipour@sku.ac.ir`

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