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A STUDY OF HORIZONTALLY WEAKLY CONFORMAL MAPS AND THEIR DISTRIBUTIONS

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Abstract. The aim of this paper is to consider horizontally weakly conformal maps which have been studied in [P. Baird and J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Mathematical Society Monographs. New Series, **29**, The Clarendon Press, Oxford University Press, Oxford, 2003]. We first generalize the results of this book for Euclidean space with arbitrary Riemannian metrics and then we study totally umbilic and totally geodesic integral manifolds of the projection map, when regarded as conformal submersion between Euclidean spaces with arbitrary Riemannian metrics.

1. **Introduction**

The aim of this paper is to consider horizontally weakly conformal maps which have been studied in [3]. Horizontally weakly conformal maps are generalization of Riemannian submersions in a sense that at the point where $d\psi_x \neq 0$, where $d\psi_x$ denotes the differential of the map $\psi : (M^n, g) \to (\overline{M}^m, h)$, $d\psi_x$ preserves horizontal angels [6]. This is equivalent to the existence of a function Λ on M such that $\langle d\psi_x(X), d\psi_x(Y)\rangle_h = \Lambda(x)\langle X, Y\rangle_g$, for any horizontal vectors X, Y.

In section 2 of this paper, we study horizontally weakly conformal maps from a Riemannian manifold into an Euclidean space equipped with an arbitrary Riemannian metric and obtain some generalizations

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of some results of [3] for Euclidean spaces with the canonical metric. We also give some results regarding the relationship between these maps and two dimensional Riemannian manifolds.

Finally, we study the horizontal and vertical distributions of the projection map between the Euclidean spaces equipped with an arbitrary Riemannian metrics, under the condition of conformality, and obtain some generalizations of the results in [3] which have been proved there in the setting of the Euclidean spaces equipped with their canonical metric.

2. Main Results

Definition 2.1. [3] *Let $\psi : (M^n, g) \to (\overline{M}^m, h)$ be a smooth map from Riemannian manifold (M, g) into Riemannian manifold $(\overline{M}, h)$. The map ψ is called horizontally weakly conformal when*

- $d\psi_x = 0$, *or*
- *the linear transformation $d\psi_x : T_xM \to T_{\psi(x)}\overline{M}$ is surjective and there is a number $\Lambda(x)$, which is called square dilation, such that for any $X, Y \in \mathcal{H}_x$,*

$$\langle d\psi_x(X), d\psi_x(Y)\rangle_h = \Lambda(x) \langle X, Y\rangle_g \,,$$

where $\mathcal{H}_x = \{\ker d\psi_x\}^\perp$ is the horizontal space.

Lemma 2.2. *Let $\psi = (\psi^1, \dots, \psi^m)$ be a smooth map from Riemannian manifold (M, g) into the Euclidean space $\mathbb{R}^m$ (with or without a Riemannian metric). Then $\ker d\psi = \cap_{i=1}^m \ker d\psi^i$ and $\mathcal{H}_\psi = \mathcal{H}_{\psi^1} + \cdots + \mathcal{H}_{\psi^m}$ which for every $i = 1, \dots, m$, $\mathcal{H}_{\psi^i} = \langle \nabla\psi^i\rangle$. In particular, if ψ is a submersion then $\mathcal{H}_\psi = \mathcal{H}_{\psi^1} \oplus \cdots \oplus \mathcal{H}_{\psi^m}$.*

Theorem 2.3. *Let $\psi = (\psi^1, \dots, \psi^m)$ be a smooth map from Riemannian manifold (M, g) into Riemannian manifold $(\mathbb{R}^m, h)$. Then ψ is a horizontally weakly conformal map with the square dilation Λ if and only if for any $i, j = 1, \dots, m$, $\langle \nabla\psi^i, \nabla\psi^j\rangle_g = \Lambda(h^{ij} \circ \psi)$.*

Proof. By Lemma 2.2 and for any $i, j = 1, \dots, m$, we have

$$\langle d\psi(\nabla\psi^i), d\psi(\nabla\psi^j)\rangle_h = \Lambda \langle \nabla\psi^i, \nabla\psi^j\rangle_g \,,$$

and so

$$\begin{aligned}\sum_{k,l=1}^m d\psi^k(\nabla\psi^i)(h_{kl} \circ \psi) d\psi^l(\nabla\psi^j) &= \sum_{k,l=1}^m \left\langle \nabla\psi^i, \nabla\psi^k\right\rangle_g (h_{kl} \circ \psi) \left\langle \nabla\psi^l, \nabla\psi^j\right\rangle_g \\ &= \Lambda \langle \nabla\psi^i, \nabla\psi^j\rangle_g \,,\end{aligned}$$

which yields the conclusion. □

Theorem 2.4. *Let $\psi = (\psi^1, \dots, \psi^m)$ be a smooth map from Riemannian manifold (M, g) into Riemannian manifold $(\mathbb{R}^m, fh_{can})$, where f is a positive smooth function on the Euclidean space $\mathbb{R}^m$.*

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Then ψ is a horizontally weakly conformal map with the square dilation Λ if and only if for any $i, j = 1, \ldots, m$, $\Lambda\delta_{ij} = (f \circ \psi) \left\langle \nabla\psi^i, \nabla\psi^j \right\rangle_g$.

Corollary 2.5. [3] *Let $\psi = (\psi^1, \ldots, \psi^m)$ be a smooth map from Riemannian manifold (M, g) into the Euclidean space $\mathbb{R}^m$. Then, ψ is a horizontally weakly conformal map with square dilation Λ if and only if for any $i, j = 1, \ldots, m$, $\Lambda\delta_{ij} = \left\langle \nabla\psi^i, \nabla\psi^j \right\rangle_g$.*

Theorem 2.6. [3] *Composition of two horizontally weakly conformal maps $\psi : (M^n, g) \to (\overline{M}^m, h)$ and $\varphi : (\overline{M}^m, h) \to (\overline{\overline{M}}^k, l)$ with square dilations $\Lambda : M \to [0, \infty)$ and $\overline{\Lambda} : \overline{M} \to [0, \infty)$, is a horizontally weakly conformal map $\varphi \circ \psi$ with square dilation $\Lambda(\overline{\Lambda} \circ \psi) : M \to [0, \infty)$.*

Two dimensional Riemannian manifolds have locally isothermal coordinates. Therefore by use of Theorems 2.4 and 2.6, we get the following result.

Theorem 2.7. *Consider $\psi : (M^n, g) \to (\overline{M}^2, h)$ is a smooth map from Riemannian manifold (M, g) into two dimensional Riemannian manifold $(\overline{M}, h)$. Then ψ is a horizontally weakly conformal map if and only if for any local isothermal coordinate z (write z in a complex form) on $\overline{M}^2$, ∇z is isotropic, that is $\langle \nabla z, \nabla z \rangle_g = 0$.*

Corollary 2.8. [3] *A smooth map from Riemannian manifold (M, g) into a Riemann surface $\overline{M}^2$, is a horizontally weakly conformal map if and only if for any local complex coordinate z on $\overline{M}^2$, ∇z is isotropic.*

Lemma 2.9. *Suppose (M^n, g) is a smooth Riemannian manifold and $(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n)$ be its local coordinate and consider distributions $\mathcal{H} = \left\langle \nabla x^1, \cdots, \nabla x^m \right\rangle$ and $\mathcal{V} = \left\langle \frac{\partial}{\partial x^{m+1}}, \cdots, \frac{\partial}{\partial x^n} \right\rangle$. Then*

- *$\mathcal{H} = \mathcal{V}^\perp$,*
- *The distribution $\mathcal{H}$ is integrable if and only if for any $i, j = 1, \ldots, m$, $r = m + 1, \ldots, n$,*

$$\sum_{A,B=1}^{n} \left(g^{iB} g^{jA} - g^{jB} g^{iA} \right) \frac{\partial g_{Ar}}{\partial x^B} = 0,$$

- *The distribution $\mathcal{V}$ is integrable,*
- *The maximal integral manifolds of $\mathcal{V}$ are totally geodesic if and only if for any $i = 1, \ldots, m$, $r, s = m + 1, \ldots, n$,*

$$\sum_{j=1}^{m} T^{ij} \Gamma^j_{rs} = 0,$$

where $\{\Gamma^j_{rs}\}$ are Christoffel symbols of Levi-Civita connection ∇ and $(T_{ij}) = (g^{ij})$,

- *The maximal integral manifolds of $\mathcal{V}$ are totally umbilic if and only if for any $i = 1, \ldots, m$, $r, s = m + 1, \ldots, n$,*

$$\sum_{j=1}^{m} T^{ij} \left(\Gamma^j_{rs} - \frac{g_{rs}}{n - m} \sum_{t,v=m+1}^{n} L^{tv} \Gamma^j_{tv} \right) = 0,$$

where $(L_{tv}) = (g_{tv})$.

Theorem 2.10. *Consider the projection map* $\psi : (\mathbb{R}^n, g) \to (\mathbb{R}^m, h)$, $\psi(x^1, \dots, x^m, x^{m+1}, \dots, x^n) = (x^1, \dots, x^m)$. *In any point* $\mathbf{x} \in \mathbb{R}^n$, *vertical space* $\mathcal{V}_{\mathbf{x}}$ *is spanned by* $\{\frac{\partial}{\partial x^{m+1}}, \cdots, \frac{\partial}{\partial x^n}\}$ *and horizontal space* $\mathcal{H}_{\mathbf{x}}$ *is spanned by* $\{\overset{g}{\nabla} x^1, \cdots, \overset{g}{\nabla} x^m\}$. *Then* ψ *is a conformal submersion with the square dilation* Λ *if and only if for any* $i, j = 1, \dots, m$, $g^{ij} = \Lambda(h^{ij} \circ \psi)$, *and so*

- $\mathcal{H} = \mathcal{V}^{\perp}$,
- *The distribution* $\mathcal{H}$ *is integrable if and only if for any* $i, j = 1, \dots, m$, $r = m+1, \dots, n$,

$$\Lambda^2 \sum_{k,l=1}^{m} \Big((h^{il}h^{jk} - h^{jl}h^{ik}) \circ \psi \Big) \; \frac{\partial g_{kr}}{\partial x^l}$$
$$+ \Lambda \sum_{s=m+1,\dots,n,\; l=1,\dots,m} \Big((h^{il} \circ \psi) \; g^{js} - (h^{jl} \circ \psi) \; g^{is} \Big) \frac{\partial g_{sr}}{\partial x^l}$$
$$+ \Lambda \sum_{s=m+1,\dots,n,\; l=1,\dots,m} \Big(g^{is}(h^{jl} \circ \psi) - g^{js}(h^{il} \circ \psi) \Big) \frac{\partial g_{lr}}{\partial x^s}$$
$$+ \sum_{t,s=m+1}^{n} \left(g^{is}g^{jt} - g^{js}g^{it} \right) \frac{\partial g_{tr}}{\partial x^s} = 0,$$

- *The distribution* $\mathcal{V}$ *is integrable and its maximal integral manifolds are* $(n-m)$*-dimensional planes which are fibers of* ψ,
- *The plane fibers are totally geodesic if and only if for any* $i = 1, \dots, m$, $r, s = m+1, \dots, n$,

$$\sum_{j=1}^{m} (h_{ij} \circ \psi) \Gamma^j_{rs} = 0,$$

  where $\{\Gamma^j_{rs}\}$ *are Christoffel symbols of Levi-Civita connection* $\overset{g}{\nabla}$,
- *The plane fibers are totally umbilic if and only if for any* $i = 1, \dots, m$, $r, s = m+1, \dots, n$,

$$\sum_{j=1}^{m} (h_{ij} \circ \psi) \Big(\Gamma^j_{rs} - \frac{g_{rs}}{n-m} \sum_{t,v=m+1}^{n} L^{tv} \Gamma^j_{tv} \Big) = 0,$$

  where $(L_{tv}) = (g_{tv})$.

Corollary 2.11. [3] *Consider the orthogonal projection* $\psi : \mathbb{R}^n \to \mathbb{R}^m$, $\psi(x^1, \dots, x^m, x^{m+1}, \dots, x^n) = (x^1, \dots, x^m)$. *In any point* $\mathbf{x} \in \mathbb{R}^n$, *vertical space* $\mathcal{V}_{\mathbf{x}}$ *is spanned by* $\{\frac{\partial}{\partial x^{m+1}}, \cdots, \frac{\partial}{\partial x^n}\}$ *and horizontal space* $\mathcal{H}_{\mathbf{x}}$ *is spanned by* $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^m}\}$, *and the map* ψ *is a conformal submersion with square dilation one. Distributions* $\mathcal{V}$ *and* $\mathcal{H}$, *are orthogonal to each other, integrable and their maximal integral manifolds are totally geodesic, and respectively are* $(n-m)$*-dimensional and* m*-dimensional planes.*

Theorem 2.12. *Consider the double covering map* $\psi : (\mathbb{S}^n, g) \to (\mathbb{R}P^n, h_{std})$, $\psi(p) = [p]$, $p \in \mathbb{S}^n$, *where* g *is a Riemannian metric on* $\mathbb{S}^n$ *and* h_{std} *is the standard Riemannian metric of* $\mathbb{R}P^n$. *Then* ψ

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is a conformal submersion if and only if g and the standard Riemannian metric of $\mathbb{S}^n$ are conformally equivalent.

3. Conclusions

In this paper, we considered the horizontally weakly conformal maps which have been studied in [3]. We generalized the results of [3] for Euclidean space with arbitrary Riemannian metrics and then we studied totally umbilic and totally geodesic integral manifolds of the projection map, when regarded as conformal submersion between Euclidean spaces with arbitrary Riemannian metrics.

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